

PARAMETRIC RESONANCE OF ELASTIC SYSTEMS WITH DISTRIBUTED PARAMETERS

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Abstract

Full Text

MATHEMATICS

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PARAMETRIC RESONANCE OF ELASTIC SYSTEMS WITH DISTRIBUTED PARAMETERS

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1. In a separable Hilbert space H , consider the Hamiltonian equation

$$i \frac{d}{dt} \mathcal{F} x = \frac{1}{\theta} [I + \varepsilon \mathcal{H}(\tau)] x, \tag{1}$$

whose coefficients satisfy the conditions:

- a) \mathcal{F} is a self-adjoint completely continuous operator having an (unbounded) inverse \mathcal{F}^{-1} .
- b) The operator function $\mathcal{H}(\tau)$ has the form

$$\mathcal{H}(\tau) = \sum_{k=-m}^m \mathcal{H}^{(k)} e^{ik\tau},$$

is a symmetric operator for each τ , and the operators $\mathcal{F}^{-1} \mathcal{H}^{(k)}$, $k = 0, \pm 1, \pm 2, \dots, \pm m$, are bounded.

The positive parameters ε and θ characterize the “amplitude” and frequency of the perturbation, and I is the identity operator in H . Equation (1) with $\varepsilon = 0$ will be called **unperturbed**.

In note ⁽¹⁾, a self-adjoint operator S_0 and a bounded linear operator S_1 , acting in a certain Hilbert space L_2 , were constructed such that the eigenvalues of the operator $S_0 + \varepsilon S_1$ are in a simple relation with the eigenvalues of the monodromy operator ⁽¹⁾ of equation (1).

Let α_0 be an arbitrary fixed point of the complex plane, δ some positive number, and Ω the set of points λ of the real axis satisfying the condition $|i\lambda - \alpha_0| \leq \delta$. Through the spectral family E_λ of the operator S_0 , define the projector Q_δ by the formula

$$Q_\delta = \int_{\Omega} dE_\lambda.$$

From Theorems 2-3 ⁽¹⁾ the following assertion follows:

For a number α satisfying the condition

$$\|(\alpha - \alpha_0)I - i\varepsilon S_1\| < \delta, \quad (2)$$

to be a characteristic exponent ⁽¹⁾ of equation (1), it is necessary and sufficient that this number be an eigenvalue of the operator

$$\Phi(\varepsilon, \delta, \alpha, \alpha_0) = iQ_\delta(S_0 + \varepsilon S_1)Q_\delta + \quad (3)$$

$$+ \varepsilon^2 Q_\delta S_1 \{I - (iS_0 - \alpha_0 I)^{-1} P_\delta [(\alpha - \alpha_0)I - i\varepsilon S_1]\}^{-1} (iS_0 - \alpha_0 I)^{-1} P_\delta S_1 Q_\delta.$$

Here I is the identity operator in L_2 , $P_\delta = I - Q_\delta$, and $(iS_0 - \alpha_0 I)^{-1} P_\delta$ is the operator inverse to the restriction of the operator $(iS_0 - \alpha_0 I)$ to the subspace $P_\delta L_2$.

For sufficiently small values of ε and $|\alpha - \alpha_0|$, it is not difficult to obtain the estimate

$$\begin{aligned} & \|\Phi(\varepsilon, \delta, \alpha, \alpha_0) - iQ_\delta(S_0 + \varepsilon S_1)Q_\delta\| \leq \\ & \leq \delta^{-1} \varepsilon^2 \|S_1\|^2 (1 - \delta^{-1} \|i\varepsilon S_1 - (\alpha - \alpha_0)I\|)^{-1}, \end{aligned}$$

from which it follows that the operator $iQ_\delta(S_0 + \varepsilon S_1)Q_\delta$, for these values of ε and $|\alpha - \alpha_0|$, is the “principal” part of the operator $\Phi(\varepsilon, \delta, \alpha, \alpha_0)$. We shall call the eigenvalues of the operator $iQ_\delta(S_0 + \varepsilon S_1)Q_\delta$ the first approximation to those characteristic exponents α of equation (1) for which condition (2) is satisfied.

In what follows we shall number the eigenvalues λ_j of the operator \mathcal{F}^{-1} in order of increasing $|\lambda_j|$, determining the sign of the index j from the condition $\text{sign } j \cdot \lambda_j > 0$. If, moreover, $\{\lambda_j\}$ is a subsequence of the eigenvalues λ_j , then for the subindex k the condition $\text{sign } k \lambda_{jk} > 0$ is assumed to be satisfied.

Theorem 1. *Let the numbers $\varepsilon, \theta, \delta$ satisfy the condition*

$$\varepsilon \theta^{-1} \max_{\tau \in [0, 2\pi]} \|\mathcal{F}^{-1} \mathcal{H}(\tau)\| < \delta < \frac{1}{2}.$$

Denote by λ_{jk} those eigenvalues of the operator \mathcal{F}^{-1} that satisfy the inequalities

$$|m_{jk} + \theta^{-1} \lambda_{jk} - i\alpha_0| \leq \delta,$$

where m_{jk} are integers uniquely determined by these inequalities. Then the characteristic exponents α of equation (1), satisfying condition (2), coincide

in the first approximation with the eigenvalues of the matrix $\Omega(\varepsilon, \delta, \alpha_0)$, whose matrix elements are the numbers

$$\Omega_{kl}(\varepsilon, \delta, \alpha_0) = -i(m_{jk} + \theta^{-1}\lambda_{jk})\delta_{lk} - i\varepsilon\delta^{-1}|\lambda_{jk}| \operatorname{sign} l (\mathcal{H}^{(m_{jk}-m_{jl})} a_{jl}, a_{jk}).$$

Here a_{jk} are orthonormal eigenvectors of the operator \mathcal{F}^{-1} corresponding to the eigenvalues λ_{jk} , $(a_{jk}, a_{jl}) = \delta_{lk}$, and $\mathcal{H}^{(N)} \equiv 0$ for $|N| > m$.

2. We shall call equation (1) **stable** if all its solutions are bounded as $\tau \rightarrow \infty$, and **unstable** otherwise. As a characteristic of the degree of instability of equation (1), it is natural to take the quantity

$$\rho(\varepsilon, \theta) = \frac{1}{2\pi} \lim_{n \rightarrow \infty} n^{-1} \ln \|X^n(2\pi, \varepsilon, \theta)\|, \quad (4)$$

where $X(2\pi, \varepsilon, \theta)$ is the monodromy operator of equation (1). By theorems 1-2 of (1), the function $\rho(\varepsilon, \theta)$ coincides with the largest real part of the characteristic exponents of equation (1).

Theorem 2. Suppose that, for a given positive number θ_0 , there exist natural numbers N_1, N_2 for which the inequalities

$$\lambda_{N_1+1} - \lambda_{N_1} > m\theta_0, \quad \lambda_{-N_2} - \lambda_{-N_2-1} > m\theta_0$$

hold. Let

$$\delta_1 = \max_{\tau \in (0, 2\pi)} \|\mathcal{F}^{-1}\mathcal{H}(\tau)\|, \quad N = \max(N_1, N_2), \quad \delta_2 = \min_{\lambda_i \neq \lambda_j} |\lambda_i - \lambda_j|, \quad i, j = \pm 1, \pm 2, \dots, \pm N,$$

and let the positive quantities $\varepsilon, \theta, \delta$ satisfy the inequalities

$$\theta^{-1}\varepsilon\delta_1 < \delta < \frac{1}{2\theta_0} \min(1, \delta_2).$$

Then for the function $\rho(\varepsilon, \theta)$, defined by formula (4), for any θ in the interval $(0, \theta_0)$ the representation

$$\rho(\varepsilon, \theta + \varepsilon\mu) = \rho_1(\varepsilon, \theta + \varepsilon\mu) + O(\varepsilon^{1+1/2N}) \quad (5)$$

is valid.

Here $\rho_1(\varepsilon, \theta)$ is the largest real part of the eigenvalues of the matrix $\Omega(\varepsilon, \delta, \alpha_j)$ (see Theorem 1), constructed from the characteristic exponents $\alpha_j = i\lambda_j\theta^{-1}$, $j = \pm 1, \pm 2, \dots, \pm N$, of the unperturbed equation, and μ is an arbitrary real number.

The estimate in (5) is uniform in μ as it varies over any finite interval.

The assertion just formulated shows that the second operator in the right-hand side of formula (3), for sufficiently small values of ε , plays in the present case the role of a correction term. This confirms the reasonableness of the concept of the first approximation for the characteristic exponents of equation (1).

3. With the aid of Theorem 1 one can obtain certain analogues of results known for the case of Hamiltonian equations with periodic matrix coefficients (2-4).

We shall call the **domain of dynamic instability of the first approximation** the set in the space of the parameters ε, θ of equation (1) defined by the inequality $\rho_1(\varepsilon, \theta) > 0$, where the function $\rho_1(\varepsilon, \theta)$ was introduced above in Theorem 2. We shall say that for $\theta = \theta_0, \varepsilon = 0$ equation (1) is in the "general" case if the following conditions are satisfied: 1) there exists a unique pair of eigenvalues λ', λ'' of the operator \mathcal{F}^{-1} satisfying the condition $\lambda' \lambda'' < 0$ and representable in the form $N\theta_0 = |\lambda' - \lambda''|$ with natural $N \leq m$; 2) for all eigenvalues λ_j commensurable with λ', λ'' modulo θ_0 , the relations $|\lambda' - \lambda_j| > m\theta_0, |\lambda'' - \lambda_j| > m\theta_0$ hold. Here m is the greatest index of the Fourier components of the operator function $\mathcal{H}(\tau)$.

Theorem 3. *Suppose that for equation (1), at $\theta = \theta_0, \varepsilon = 0$, the "general" case obtains.*

Then the domain of dynamic instability of the first approximation, adjacent for sufficiently small ε to the point $(0, \theta_0)$, is given by the inequalities

$$\theta_1(\varepsilon) < \theta < \theta_2(\varepsilon), \quad (6)$$

where

$$\begin{aligned} \theta_{1,2} &= \theta_0 + \varepsilon\mu_{1,2}; & \mu_{1,2} &= N^{-1} [\gamma_{11} - \gamma_{22} \pm \sqrt{-\gamma_{12}\gamma_{21}}]; \\ \gamma_{11} &= \lambda'(\mathcal{H}^{(0)}a', a'); & \gamma_{22} &= \lambda''(\mathcal{H}^{(0)}a'', a''); \\ \gamma_{12} &= \lambda'(\mathcal{H}^{(N)}a', a''); & \gamma_{21} &= \lambda''(\mathcal{H}^{(-N)}a'', a'); \end{aligned} \quad (7)$$

$$N = \theta_0^{-1}(\lambda' - \lambda''); \quad \lambda' \lambda'' < 0$$

and a', a'' are the eigen-elements of the operator \mathcal{F}^{-1} for the eigenvalues λ', λ'' , respectively.

It follows from formulas (6)–(7) that if $\mathcal{H}^{(0)} = 0$, then the width $|\mu_1 - \mu_2|$ of the instability domain of the first approximation is proportional to the quantity

$$|N^{-1} \sqrt{-\gamma_{12}\gamma_{21}}| = |N^{-1} \sqrt{|\lambda' \lambda''|} |(\mathcal{H}^{(N)}a', a'')|. \quad (8)$$

Consequently, in this case the perturbation frequencies θ can be classified according to the degree of "danger" in accordance with the values of the quantities (8).

Let $\varepsilon(\alpha)$ denote the exact upper bound of the values of ε such that, in the instability domain, for the general solution $x(\tau)$ of equation (1), the estimate

$$\|x(\tau)\| = O(e^{\alpha\tau}), \quad \alpha > 0 \quad (9)$$

is valid. Then for the function $\varepsilon(\alpha)$, by means of Theorem 1, in the “general” case it is not difficult to obtain the relation

$$\varepsilon(\alpha) = \alpha\theta/\sqrt{-\gamma_{12}\gamma_{21}} + O(\alpha^2).$$

Thus, as in the finite-dimensional case ^(2,3), the greater the quantity (8), the wider, on the one hand, is the domain of dynamic instability of the first approximation adjacent to the point $\varepsilon = 0$, $\theta_0 = N^{-1}|\lambda' - \lambda''|$, and, on the other hand, the closer to the θ -axis comes the domain for which the estimate (9) is violated for fixed sufficiently small $\alpha > 0$.

4. **Example.** The problem of the dynamic stability of a thin-walled rod having two axes of symmetry and loaded by periodic couples leads to a system of two differential equations for the lateral displacement $u(z, t)$ and the angle of twist $\varphi(z, t)$ ⁽⁵⁾

$$EJ_y \frac{\partial^4 u}{\partial z^4} + (M_0 + M_1 \cos \theta t) \frac{\partial^2 \varphi}{\partial z^2} + m \frac{\partial^2 u}{\partial t^2} = 0,$$

$$EJ_\omega \frac{\partial^4 \varphi}{\partial z^4} + (M_0 + M_1 \cos \theta t) \frac{\partial^2 u}{\partial z^2} - GJ_d \frac{\partial^2 \varphi}{\partial z^2} + mp^2 \frac{\partial^2 \varphi}{\partial t^2} = 0. \quad (10)$$

The values of the coefficients entering here can be found in ⁽⁵⁾.

Along with equations (10), boundary conditions are specified, determined by the conditions of fastening of the rod at its ends. The fastening conditions may be of the following type: 1. Hinged support: at the corresponding end $u = \varphi = \partial^2 u / \partial z^2 = \partial^2 \varphi / \partial z^2 = 0$. 2. Complete clamping $u = \varphi = \partial u / \partial z = \partial \varphi / \partial z = 0$. 3. Free end $\partial^2 u / \partial z^2 = \partial^3 u / \partial z^3 = \partial^2 \varphi / \partial z^2 = 0$, $\partial^3 \varphi / \partial z^3 = (GI_d / EI_\omega) \partial \varphi / \partial z$. As boundary conditions, the combinations (conditions at different ends) 1—1, 1—2, 2—2, 2—3 are possible.

The system of equations (10), with account taken of the boundary conditions, is reduced to the operator Hamiltonian equation (1).

Denote by ω_{2j-1} and ω_{2j} , $j = 1, 2, \dots$, respectively, the frequencies of the flexural and torsional vibrations of the unloaded rod. Then, with the aid of Theorem 1, it can be shown that the regions of dynamic instability of the first approximation can be adjacent only to excitation frequencies of the form

$$\theta = \theta(j_1, j_2) = \omega_{2j_1-1} + \omega_{2j_2}. \quad (11)$$

For the instability region itself, for sufficiently small values of the quantity $|M_0| + |M_1|$, adjacent to frequency (11), the following formulas can be obtained with the aid of Theorem 3:

$$\theta_-(j_1, j_2) < \theta < \theta_+(j_1, j_2), \quad (12)$$

where

$$\theta_{\pm}(j_1, j_2) = \theta(j_1, j_2) \pm \frac{|M_1|}{2\sqrt{\omega_{2j_1-1}\omega_{2j_2}}} \cdot \frac{1}{m\rho l} \left| \int_0^l \frac{d\varphi_{2j_1-1}}{dz} \frac{d\varphi_{2j_2}}{dz} dz \right|,$$

l is the length of the rod, and $\varphi_{2j-1}(z)$, $\varphi_{2j}(z)$, $j = 1, 2, \dots$, are the flexural and torsional modes of vibration of the rod corresponding to the eigenfrequencies ω_{2j-1} , ω_{2j} , respectively.

It follows from formulas (11)–(12) that, in the system under consideration, the loss of stability must be accompanied by intense flexural-torsional vibrations (2).

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- ¹ V. N. Fomin, DAN, **163**, No. 4 (1965).
- ² V. A. Yakubovich, DAN, **121**, No. 4 (1958).
- ³ M. G. Krein, V. A. Yakubovich, Transactions of the International Symposium on Nonlinear Oscillations, 1, Kiev, 1963.
- ⁴ I. G. Malkin, Some Problems in the Theory of Nonlinear Oscillations, Moscow, 1956.
- ⁵ V. V. Bolotin, Dynamic Stability of Elastic Systems, Moscow, 1956.

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