ON IRREDUCIBLE EXCEPTIONAL SUBMANIFOLDS OF THE FIRST KIND ON THREE-DIMENSIONAL COMPLEX-ANALYTIC MANIFOLDS

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Abstract

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ON IRREDUCIBLE EXCEPTIONAL SUBMANIFOLDS OF THE FIRST KIND ON THREE-DIMENSIONAL COMPLEX-ANALYTIC MANIFOLDS

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In the theory of algebraic surfaces, an **exceptional curve of the first kind** is a curve that can be contracted to a nonsingular point. The Castelnuovo-Enriques theorem, generalized to the analytic case by Grauert, asserts that for an irreducible curve S on a nonsingular surface F to be an exceptional curve of the first kind, it is necessary and sufficient that S be a nonsingular rational curve and $S^2 = -1$.

Definition. Let V be a complex-analytic manifold, $\dim V = 3$, and let D be an analytic subspace on V. We shall call D an **exceptional submanifold of the first kind on** V if there exist a three-dimensional manifold V', a submanifold (nonsingular subspace) D' on it, and a regular proper mapping $T : V \to V'$ such that $T(D) = D'$, T establishes a biregular correspondence between $V - D$ and $V' - D'$, and the mapping $T|_D : D \to D'$ is not biregular.

Theorem 1. *Let* V *be a complex-analytic manifold,* $\dim V = 3$*, and let* D *be an irreducible subspace on* V *which is an exceptional submanifold of the first kind* (V', D', T *as in the definition).*

Then, if on V' *one performs the* σ*-process along* D' *and denotes the embedded submanifold by* $\overline{D}$*, the object obtained from* V' *by* $\overline{V}$*, and the corresponding regular mapping* $\overline{V} \to V'$ *by* $\overline{T}$*, there exists a biregular correspondence between* $\overline{V}$ *and* V*, under which* $\overline{D}$ *coincides with* D*, and* $\overline{T}$ *with* T*.*

Theorem 2. *Let* V *be a complex-analytic manifold,* $\dim V = 3$*, and let* D *be a compact nonsingular ruled surface on* V*, more precisely, a fibration into projective lines with base a compact nonsingular curve. Introduce the notation:* l *is a fiber of the fibration;* C *is the base;* $\pi : D \to C$ *is the projection of the fibration onto the base. Let* $D \cdot l = -1$ *(intersection index on* V*).*

Then there exists a complex-analytic manifold V'*, on it a curve* D'*, biregularly equivalent to the curve* C*, and a regular mapping* $T : V \to V'$ *such that* T *establishes a biregular correspondence between* $V - D$ *and* $V' - D'$*, and* $T|_D : D \to D'$ *coincides with* $\pi : D \to C$*.*

Using Theorems 1 and 2, and also the Kodaira–Grauert condition ([1,2]) for contractibility of a projective space to a nonsingular point, we obtain the following.

Criterion. For an irreducible compact subspace D on a three-dimensional complex-analytic manifold V to be an exceptional submanifold of the first kind, it is necessary and sufficient that D be either a projective plane with $D \cdot e = -1$, where e is a line on D, or a fibration into projective lines with nonsingular compact curve C as base and with $D \cdot l = -1$, where l is a fiber of the fibration. In the first case the contraction coincides with the anti-σ-process to a nonsingular point, and in the second with the anti-σ-process onto a nonsingular curve. (By an anti-σ-process we here mean a mapping inverse to a σ-process.)

The question arises: if V is an algebraic variety, will the variety V', obtained after contracting the exceptional subvariety D mentioned in the criterion, always be algebraic?

For contraction to a point this follows from Kodaira's result [1]. As for contraction to a curve, it follows from Hironaka' s results [3] that V' may be neither algebraic nor even an abstract algebraic variety.

Indeed, on the one hand, examples are known of 3-dimensional complex-analytic varieties with three algebraically independent meromorphic functions which are not algebraic varieties (neither projective nor abstract); on the other hand, Hironaka proved that any such variety can, by a sequence of monoidal transformations with nonsingular centers, be made into a projective algebraic variety.

A rather direct example showing that, under contraction to a curve, a non-algebraic variety can be obtained from an algebraic variety is constructed as follows.

In projective space P^4 one must take a hypersurface F_n of degree $n > 2$ having one quadratic nondegenerate singular point O (all other points being nonsingular). Then the σ-process on P^4 at the point O transforms F_n into a nonsingular variety $\bar{F}_n$, and a quadric $(P^1 \times P^1)$ is glued into F_n, which has intersection index -1 with each of its line factors. If it is contracted onto either of these factors, then from $\bar{F}_n$ one obtains a variety $\bar{\bar{F}}_n$ which is neither projective, nor abstract algebraic, nor a Kähler variety.

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[1] K. Kodaira, Ann. Math., **60**, 28 (1954).
[2] H. Grauert, Math. Ann., **146**, 331 (1962).
[3] H. Hironaka, Ann. Math., **79**, No. 1 (1964).

Note: Figure translations are in progress. See original paper for figures.