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Abstract

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MATHEMATICS

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ON CRITERIA OF COMPACTNESS IN SPACES OF ABSTRACT FUNCTIONS AND ON COMPLETE CONTINUITY OF THE EMBEDDING OPERATOR

(Presented by Academician S. L. Sobolev on 16 VII 1964)

Definition 1 ⁽¹⁾. **1)** Denote by $\sum(R_n)$ the collection of all L -measurable subsets of finite measure of Euclidean space R_n . Denote by Λ the collection of all finite linear combinations of characteristic functions of sets $E \in \sum(R_n)$. By $\Phi_p(X, R_n)$, $1 < p < \infty$, we denote the collection of all additive set functions $\varphi(E)$, defined on $\sum(R_n)$, taking values in the B -space X and having finite norm

$$\|\varphi\|_{\Phi_p(X, R_n)} = \sup_{\omega \in \Lambda} \left\| \int_{R_n} \omega d\varphi \right\|_X / \|\omega\|_{L_{p'}}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (1)$$

2) A function φ from Φ_p is said to belong to $\Psi_p(X, R_n)$ if

$$\lim_{|h| \rightarrow 0} \|\varphi(E+h) - \varphi(E)\|_{\Phi_p} = 0. \quad (2)$$

3) In what follows, by Ω we shall everywhere mean a bounded simply connected domain in R_n . One says that $\varphi \in \Phi_p(X, \Omega)$ (respectively $\varphi \in \Psi_p(X, \Omega)$) if $\varphi \in \Phi_p(X, R_n)$ (respectively $\varphi \in \Psi_p(X, R_n)$) and $\varphi(E) = \varphi(E \cap \Omega)$, $E \in \sum(R_n)$.

For every function $\omega = \sum \alpha_i \chi_{E_i}$ from Λ , by definition we put

$$\int \omega d\varphi = \sum \alpha_i \varphi(E_i).$$

In view of (1),

$$\left\| \int \omega d\varphi \right\| \leq \|\omega\|_{L_{p'}} \|\varphi\|_{\Phi_p}. \quad (3)$$

Using (3) and the density of Λ in $L_{p'}$, one may, by a limiting passage, define for any function $g \in L_{p'}$ the integral $\int g d\varphi$, which makes it possible, by means of a known integral identity, to introduce for some functions from Φ_p the notion of generalized derivative ⁽¹⁾ and thereby to construct analogues of the spaces W_p^l ⁽²⁾ and H_p^r ⁽³⁾—the spaces $\Phi_p^l(X, \Omega)^*$ and $H_p^r(X, \Omega)$ ⁽⁴⁾, for which ^(1, 4, 5) the same embedding theorems are valid as for W_p^l, H_p^r .

The specific feature of these embedding theorems is that, in the case of infinite-dimensional X , the embedding operator will not be completely continuous.

Example 1. Let A be a bounded noncompact set in X . Consider the functions $\varphi_x(E) = xmE, x \in A, E \in \sum(\Omega)$. It is clear that $D^\alpha \varphi_x = 0$ for $|\alpha| \neq 0$. Consequently,

$$\|\varphi_x\|_{\Phi_p^l} = m^{1/p} \Omega \|x\|.$$

The trace of the function φ_x under the embedding $\Phi_q^l(X, \Omega)$ into $\Phi_q^l(\Omega_s)$ has the form $\tilde{\varphi}_x(e) = xme, e \in \sum(\Omega_s)$, and

$$\|\tilde{\varphi}_x\|_{\Phi_q^{l_1}} = m^{1/q} \Omega_s \|x\|;$$

hence it follows that the set $\tilde{\Phi} = \{\tilde{\varphi}_x : x \in A\}$ is noncompact in $\Phi_q^{l_1}(X, \Omega_s)$.

* The space Φ_p^l for integer l was defined in ⁽¹⁾, and for noninteger l in ⁽⁴⁾.

The main result of the article is contained in Theorem 2 and is based on the lemmas given below.

Lemma 1. *In order that a set $F\{f\}$ from the space $C(X, \Omega)$ of continuous abstract functions be compact, it is necessary and sufficient that:*

1) *the set*

$$\Xi = \{f(y); y \in \Omega, f \in F\}$$

be compact in X ;

2) *the functions $f \in F$ be equicontinuous.*

The necessity of conditions 1) and 2) is obvious. The proof of their sufficiency can be carried out by analogy with the proof of Arzelà's theorem given in ⁽⁶⁾, p. 519. The example of the set $f_x(t) = x, x \in A$, where A is the set of

Example 1, shows that condition 1) cannot, in the case of infinite-dimensional X , be replaced by the condition that the set F be bounded in $C(X, \Omega)$.

Lemma 2 ⁽⁵⁾, p. 31. *The set of values of a function φ from $\Psi_p(X, \Omega)$ is compact in X .*

Corollary. *If $\varphi \in \Psi_p(X, \Omega)$ and $h(y)$ is a continuous function, then the set of values of the function*

$$\tilde{\varphi}(E) = \int_E h(y) d\varphi$$

is compact in X .

Lemma 3. *For compactness of a set $\Phi\{\varphi\}$ from $\Psi_p(X, \Omega)$ it is necessary and sufficient that:*

1) *the set*

$$\Xi = \{\varphi(E); E \in \sum(\Omega), \varphi \in \Phi\}$$

be compact in X ;

2) $\|\varphi\|_{\Phi_p} \leq K$ *for all $\varphi \in \Phi$;*

3) *condition 2) be satisfied uniformly with respect to φ from Φ .*

Proof. The necessity of conditions 2), 3) is obvious. The necessity of condition 1) follows from the existence of a finite $\varepsilon/2$ -net in Φ and from Lemma 2.

Sufficiency. Denote by $K_h(y)$, $h > 0$, the n -dimensional cube with center at the point y and edge $2h$, and by Ω_h the set of all points $y \in R_n$ whose distance from Ω is not more than h . Consider the functions

$$\varphi_h(y) = (2h)^{-n} \varphi(K_h(y)), \quad \varphi \in \Phi.$$

Condition 1) of Lemma 3 ensures the fulfillment of condition 1) of Lemma 1. The equicontinuity of the functions φ_h ($\varphi \in \Phi$) follows from the estimate, easily verified with the aid of (3),

$$\|\varphi_h(y_1) - \varphi_h(y_2)\|_X \leq K(2h)^{-n} m^{1/p'} (K_h(y_1) \setminus K_h(y_2)).$$

Thus, by Lemma 1 the family of functions φ_h ($\varphi \in \Phi$) is compact in $C(X, \Omega_h)$. Consequently, the family of functions of sets

$$\varphi_h(E) = \int_E \varphi_h(y) dy, \quad E \in \sum(\Omega),$$

is compact in $\Psi_p(X, \Omega)$. Using (1), pp. 314-316, one can obtain the following inequality*

$$\|\varphi_h(E) - \varphi(E)\|_{\Phi_p} \leq (2h)^{-n} \int_{K_h(0)} \|\varphi(E+y) - \varphi(E)\|_{\Phi_p} dy. \quad (4)$$

From (4), condition 2), and the compactness of the family $\varphi_h(E)$, it follows that the set Φ is compact. The set of functions of Example 1 shows that condition 1) cannot be discarded in the case of infinite-dimensional X .

Lemma 4. Let $\varphi'_{x_i} \in \Psi_p(X, \Omega)$ be the generalized derivative of a function $\varphi \in \Phi_p(X, \Omega)$. Then

$$\left\| \frac{\varphi(E + h_i) - \varphi(E)}{|h_i|} - \varphi'_{x_i}(E) \right\|_{\Phi_p} \leq \frac{1}{|h_i|} \int_0^{|h_i|} \|\varphi'_{x_i}(E + \xi_i) - \varphi'_{x_i}(E)\|_{\Phi_p} d\xi_i. \quad (5)$$

* This inequality can also be obtained with the aid of the isometry of Φ_p , proved in (4,5), to a certain subspace of the space of linear bounded operators acting from X^* into L_p .

Definition 2. Let $r = \bar{r} + \alpha$, $\bar{r} > 0$ an integer, $0 < \alpha < 1$. We say that $\varphi \in H_p^r(X, R_n)$ if $\varphi \in \Phi_p(X, R_n)$, has all unmixed generalized derivatives up to order \bar{r} from $\Phi_p(X, R_n)$, and the norm is finite

$$\|\varphi\|_{H_p^r(X, R_n)} = \|\varphi\|_{\Phi_p} + \max_{i=1,2,\dots,n} \sup_{h_i} \frac{\|\varphi_{x_i}^{\bar{r}}(E + h_i) - \varphi_{x_i}^{\bar{r}}(E)\|_{\Phi_p}}{|h_i|^\alpha}.$$

Theorem 1. If a set Φ in $H_p^r(X, R_n)$ satisfies the conditions:

- 1) the set $\Xi = \{\varphi(E); E \in \sum(R_n), \varphi \in \Phi\}$ is compact in X ;
- 2) $\|\varphi\|_{H_p^r(X, R_n)} \leq K$ for all $\varphi \in \Phi$;
- 3) $\varphi(E) = \varphi(E \cap \Omega)$ for all $\varphi \in \Phi$;
- 4) uniformly with respect to $\varphi \in \Phi$,

$$\|\varphi_{x_i}^{\bar{r}}(E + h_i) - \varphi_{x_i}^{\bar{r}}(E)\|_{\Phi_p} = o(|h_i|^\alpha), \quad i = 1, 2, \dots, n, \quad (6)$$

then Φ is compact in $H_p^r(X, R_n)$.

Proof. Using conditions 2) and 3), one can show that for all $\varphi \in \Phi$

$$\|\varphi_{x_i}^k\|_{\Phi_p} \leq K_1, \quad i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots, \bar{r}. \quad (7)$$

Hence, and from Lemma 4, we obtain

$$\begin{aligned} \|\varphi_{x_i}^k(E + h_i) - \varphi_{x_i}^k(E)\|_{\Phi_p} &\leq K_1|h_i|; \\ i = 1, 2, \dots, n; \quad k = 0, 1, \dots, \bar{r} - 1; \end{aligned} \quad (8)$$

$$\lim_{|h_i| \rightarrow 0} \left\| \frac{1}{|h_i|} (\varphi_{x_i}^k(E + h_i) - \varphi_{x_i}^k(E)) - \varphi_{x_i}^{k+1}(E) \right\|_{\Phi_p} = 0,$$

$$k = 0, 1, 2, \dots, \bar{r} - 1, \quad (9)$$

uniformly with respect to $\varphi \in \Phi$. Using condition 1) and (9) ((6) for $k = \bar{r}$), it is not difficult to show successively that the functions of the families Φ and $\Phi_{ik} = \{\varphi_{x_i}^k; \varphi \in \Phi\}$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots, \bar{r}$, satisfy condition 1) of Lemma 3, whence, together with (7), (8) ((6) for $k = \bar{r}$), by virtue of Lemma 3, it follows that the families Φ and Φ_{ik} are compact in $\Psi_p(X, \Omega)$. Thus, from any infinite subset of Φ one can select a subsequence φ_k , converging in the norm Φ_p together with the derivatives of order \bar{r} . Consider the expression

$$|h_i|^{-\alpha} \left\| (D_{x_i}^{\bar{r}} \varphi_k(E + h_i) - D_{x_i}^{\bar{r}} \varphi_l(E + h_i)) - (D_{x_i}^{\bar{r}} \varphi_k(E) - D_{x_i}^{\bar{r}} \varphi_l(E)) \right\|_{\Phi_p}. \quad (10)$$

By virtue of (6), for any $\varepsilon > 0$ one can choose $\delta > 0$ so that (10) is less than $\varepsilon/2$ for $|h_i| < \delta$. Using the invariance of the norm Φ_p with respect to shifts and the convergence of $D_{x_i}^{\bar{r}} \varphi_k$ in the norm Φ_p , one can further, for all $|h_i| \geq \delta$, choose a number N such that for $k > N$, $l > N$, (10) also becomes less than $\varepsilon/2$.

****Corollary**.** The set Φ of Theorem 1 is compact in $H_p^{r'}(X, R_n)$, $r' < 2$.

Denote by $B_1 \xrightarrow{V} B_2$ the embedding of a B -space B_1 into a B -space B_2 with bounded embedding operator V , and by VM the image of a set $M \subset B_1$ under this embedding.

Theorem 2. Let Φ be a bounded set in $\Phi_p^r(X, \Omega_n)$, satisfying condition 1) of Theorem 1, and let $\Phi_p^r(X, \Omega_n) \xrightarrow{V}$

* Cf. for $X = R_1$ with (3), p. 81.

$\rightarrow \Phi_{p_1}^{r_1}(X, \Omega_{n_1})$, where $r_1 < r^* = r - \frac{n}{p} - \frac{n_1}{p_1}$ and Ω_{n_1} is a strictly interior subdomain of the n_1 -dimensional Ω_n .

Then Φ is compact in $\Phi_{p_1}^{r_1}(X, \Omega_{n_1})$.

An analogous theorem is valid for H -classes.

Proof. Let Ω_n'' be a strictly interior subdomain of Ω_n containing Ω_n' . A function $\varphi \in \Phi_p^r(X, \Omega_n)$ can be extended ⁽⁵⁾ from Ω_n' to R_n with preservation of the class, and the extension operator is linear, bounded, and has the form

$$P\varphi = \tilde{\varphi}(I) = \int_I H(y) d\varphi, \quad I \in \sum(R_n), \quad (11)$$

where $H(y)$ is a certain continuous function equal to 1 in Ω_n and finite in Ω_n' , $\Omega_n' \subset \Omega_n'' \subset \Omega_n$. From the corollary to Lemma 2 it follows that the set of values of all functions $\tilde{\varphi}$, $\varphi \in \Phi$, is compact in X . In ⁽⁵⁾ it was shown that

$$\Phi_p^r(X, R_n) \rightarrow H_p^{r-\varepsilon}(X, R_n) \rightarrow H_p^{r-2\varepsilon}(X, R_n) \rightarrow \Phi_p^{r-3\varepsilon}(X, R_n).$$

Hence, by virtue of the corollary to Theorem 1, the set $\tilde{\Phi} = \{\tilde{\varphi}; \varphi \in \Phi\}$ is compact in $H_p^{r-2\varepsilon}$, and consequently also in $\Phi_p^{r-3\varepsilon}$. Since, by hypothesis, $r < r^*$, for sufficiently small $\varepsilon > 0$

$$\Phi_p^{r-3\varepsilon}(X, \Omega_n) \xrightarrow{V_\varepsilon} \Phi_{p_1}^{r_1}(X, \Omega_{n_1}')$$

and, moreover, $V_\varepsilon \tilde{\Phi}$ is compact in $\Phi_{p_1}^{r_1}(X, \Omega_{n_1}')$. But, by the properties of the function $H(x)$, $V\Phi = V_\varepsilon \tilde{\Phi}$. The theorem is proved.

Remark 1. If there exists an extension (11) from the domain Ω_n itself, then in Theorem 2 one may put $\Omega_{n_1}' = \Omega_{n_1}$.

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