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Abstract

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ON RHEOLOGICAL RELATIONS

IN THE MOTION OF AN ELASTIC-VISCOUS MEDIUM

IN A FIELD OF A CONSTANT LONGITUDINAL VELOCITY GRADIENT

(Presented by Academician V. A. Kargin, 21 XI 1964)

V. A. Kargin and T. I. Sogolova (1), studying the stretching of polyisobutylene, discovered an increase in viscosity with increasing specimen length. This effect was explained by the orientation of macromolecules in the direction of stretching. It is of great importance for elucidating the principal features of the processes occurring in the formation of fibers and films. An attempt at a theoretical analysis of the phenomenon on the basis of a model consisting of a suspension of rigid noninteracting ellipsoids in a viscous liquid (with allowance for Brownian motion) was undertaken by Ziabicki and Takserman-Krozer (2). Their calculations showed that in this model a steady flow is possible, and that with increasing longitudinal velocity gradient R the viscosity of the suspension may increase. In papers (3) these same authors carried out a qualitative investigation of the influence of rheological characteristics on the process of fiber formation. Coleman and Noll (4) were the first to consider the stretching of an elastic-viscous liquid from the standpoint of continuum mechanics. They applied the rheological equations they had proposed earlier to the process of steady motion of a liquid with a stagnation point, taking into account the dependence of the material parameters on the invariants of the kinematic tensors, and showed that the viscosity may increase with increasing R. Similar results were obtained in White' s work (5). However, the literature lacks a general approach to the solution of mechanical problems of fiber and film formation. Below, on the basis of rheological equations describing certain ideal elastic-viscous and thixotropic elastic-viscous bodies satisfying Oldroyd' s principle of space-time invariance (6), an analysis is made of the behavior of an elastic-viscous medium in a field of a constant longitudinal velocity gradient. Thus it is possible to show that, knowing the rheological parameters of an elastic-viscous medium characterizing it under conditions of shear deformation, one can predict how it will behave in stretching. Depending on the properties of the medium and the modes of motion, steady stretching regimes are realized with a viscosity increasing with increasing R (in the region of its low values), and a transition of the liquid (if R exceeds some critical value) into the state of a quasi-solid body undergoing brittle fracture. Cases are also possible in which, for no values of R, can established deformation regimes be realized under stretching. These results differ substantially from what is known from papers ([2–5]).

The general rheological equations of an incompressible elastic–viscous medium, with allowance for reversible structural changes, have the form ([7])

$$p'^{ik}(x^j, t) = 2\int_S^{\infty} N(s)\, ds \int_{-\infty}^{t} e^{-s(t-t')} \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \dot{\gamma}^{mn}(x'^j, t')\, dt'; \tag{1}$$

$$p'^{ik} = p^{ik} + g^{ik} p; \tag{2}$$

$$\varphi(S) = \sqrt{2S} \int_{-\infty}^{t} e^{-(t-t')S} \dot{\Gamma}(x^k, t')\, dt'. \tag{3}$$

Here x^i is an arbitrary Eulerian coordinate system; g^{ik} are the contravariant components of the fundamental metric tensor; p is the isotropic pressure, $p^{ki} = p^{ik}$ are the components of the stress tensor; $\dot{\gamma}^{mn} = \frac{1}{2}(\nabla^m v^n + \nabla^n v^m)$ are the components of the rate-of-deformation tensor; ∇^m is the operator of contravariant differentiation; v^m are the components of the velocity vector; x'^k are displacement functions satisfying the equations

$$\partial x'^k / \partial t + v^m \partial x'^k / \partial x^m = 0 \tag{4}$$

and the Cauchy conditions $x'^k(x^i, t, t')\big|_{t'=t} = x^k$. $N(s)$ is the frequency relaxation function; $S(x^k, t)$ is the function of variation of the frequency relaxation spectrum; $\varphi(s)$ is the thixotropy function, defined for $s \geq 0$, a function increasing and not convex; moreover, as shown in (7), it is determined uniquely from a shear experiment from the functions $N(s)$ and $\eta\,(\dot{\gamma}_0)$, the effective (shear) viscosity, where $\dot{\gamma}_0$ is the shear rate in steady flow. Equations (1)–(4), together with the equations of motion of a continuous medium in stresses

$$\rho(\partial v^k / \partial t + v^m \nabla_m v^k) = F^k + \nabla_m p^{km} \tag{5}$$

(F^k are the components of the body force) and incompressibility

$$\nabla_k v^k = 0 \tag{6}$$

constitute a complete system of equations describing the isothermal motion of a fluid. Let us also note that in equations (1)–(6) the usual summation convention over twice repeated indices is adopted.

Introduce Cartesian coordinates $x^1 = x,\ x^2 = y,\ x^3 = z$, and, for simplicity, consider the plane motion of an elastico-viscous fluid with a stagnation point $x = 0,\ y = 0$ under the action of a longitudinal velocity gradient $R(t) = RJ(t)$ (here $J(t)$ is the unit Heaviside function, $R = \text{const} > 0$). Motion of this kind occurs, to a first approximation, in uniaxial stretching in the direction of the x-axis of an elastico-viscous film extending without bound in the direction of the z-axis. In the case under consideration of plane deformation, equation (6) takes the form

$$\partial v_x/\partial x + \partial v_y/\partial y = 0, \tag{7}$$

and the velocity components satisfying (7) are

$$v_x = Rx, \qquad v_y = -Ry, \qquad v_z = 0. \tag{8}$$

The solution of equations (4), taking into account the Cauchy conditions, is

$$x' = xe^{-R(t-t')}, \qquad y' = ye^{R(t-t')}, \qquad z' = z. \tag{9}$$

The matrix of the rate-of-deformation tensor has the form

$$\|\dot\gamma^{ik}\| = \|\dot\gamma_{ik}\| = J(t)\begin{pmatrix} R & 0 & 0 \\ 0 & -R & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

and the second invariant of the rate-of-deformation tensor is equal to

$$\dot\Gamma = |\dot\gamma^{mn}\dot\gamma_{mn}|^{1/2} = \sqrt{2}R. \tag{11}$$

Substituting (8), (9), and (11) into (1) and (3), after simple transformations we determine the quantities $p'^{ik} = p'_{ik}$:

$$p'_{xx} = 2R\int_{S(t)}^{\infty}\frac{N(s)}{2R-s}\left[e^{(2R-s)t} - 1\right]ds,$$

$$p'_{yy} = -2R\int_{S(t)}^{\infty}\frac{N(s)}{2R+s}\left[1 - e^{-(2R+s)t}\right]ds, \tag{12}$$

$$p'_{zz} = 0, \qquad p'_{xy} = 0, \qquad p'_{yz} = 0, \qquad p'_{zx} = 0.$$

The condition of thixotropic destruction–restoration of the structure is

$$\varphi(S(t)) = 2R(1 - e^{-S(t)}). \tag{13}$$

Taking into account that in the case under consideration $g^{ik} = g_{ik} = \delta_{ik}$ (δ_{ik} is the Kronecker symbol), we obtain

$$p_{xx} = -p + p'_{xx}, \qquad p_{yy} = -p + p'_{yy}. \tag{14}$$

Substituting (12) and (14) into (5), with allowance for $F_x = -g\rho$ (g is the acceleration of gravity), $F_y = 0$, $F_z = 0$, after simple calculations we obtain the distribution of the isotropic pressure in the medium

$$p = p_0(t) - \frac{1}{2}\rho R^2(x^2 + y^2) - g\rho x, \tag{15}$$

where the quantity $p_0(t)$ must be determined from the solution of the corresponding hydrodynamic problem. For example, in the case of uniaxial stretching of a liquid film, the quantity p_{yy} is determined only by the dynamic (inertial) part of the isotropic pressure. In this case

$$p_{yy} = \frac{1}{2}\rho R^2(x^2 + y^2) + \rho g x. \tag{16}$$

Then the quantity $p_0(t)$ is determined from the second equality (12). Leaving aside the question of determining the thickness and length of the film, which is very simply solved from kinematic considerations, let us consider the rheological dependences (12) and (13).

1°. For the case of a Maxwellian fluid we shall have

$$\varphi(S) \equiv 0 \quad (S = 0), \qquad N(s) = G\delta(s - 1/\theta). \tag{17}$$

Here G is the elastic (Hookean) modulus; $\theta = \eta/G$ is the relaxation time; η is the viscosity; $\delta(x)$ is the delta function. Substituting expression (17) into (12), we shall have

$$p'_{xx} = 2RG[e^{(2R-1/\theta)t} - 1]/(2R - 1/\theta);$$

$$p'_{yy} = -2RG[1 - e^{-(2R+1/\theta)t}]/(2R + 1/\theta). \tag{18}$$

It follows from (18) that for sufficiently slow motion, when $R < 1/2\theta$, there is a steady flow with a stagnation point; in this case the expressions (18) as $t \to \infty$ take the form

$$p'_{xx}(\infty) = 2RG/(1/\theta - 2R), \qquad p'_{yy} = 2RG/(1/\theta + 2R). \tag{19}$$

The viscosity, defined by analogy with the viscosity for a Newtonian fluid,

$$\eta(R) = p'_{xx}/2R = G/(1/\theta - 2R) \tag{20}$$

increases with increasing rate of deformation. In the case $R \geq 1/2\theta$, no steady flow regime exists. In this case the stress p_{xx} grows without bound, i.e., the material behaves like a solid body and will undergo brittle fracture.

2°. Let $\varphi(S) = 0$, and let the function $N(s)$ be continuous on some interval (s_1, s_2), where $s_1 \geq 0$, and vanish outside this interval (here $s_2 = \infty$ is possible), i.e., the case considered is that of motion of a linear elastic-viscous medium in the presence of large deformations. In this case formulas (12) take the form

$$p'_{xx} = 2R\int_{s_1}^{s_2} \frac{N(s)}{2R - s}\,[e^{(2R-s)t} - 1]\,ds, \qquad p'_{yy} = -2R\int_{s_1}^{s_2} \frac{N(s)}{2R + s}\,[1 - e^{-(2R+s)t}]\,ds. \tag{21}$$

It follows hence that, just as in the case of a Maxwellian fluid, for sufficiently small $R < s_1/2$ there exists a steady flow with

viscosity

$$\eta(R) = \int_{s_1}^{s_2} \frac{N(s)\,ds}{s - 2R}, \tag{22}$$

increasing with increasing R. For $R \geq s_1/2$ the material behaves as a quasi-solid body. This case is always realized when there is a complete relaxation spectrum, if $s_1 = 0$.

3°. Let us now consider the case of an elastico-viscous thixotropic medium, when $\varphi(s) \neq 0$. In this case it is necessary to consider equations (12) jointly with the implicit equation (13), from which the function $S(t)$ is uniquely determined (see (7)). We note that from (13) it follows that, as $t \to \infty$,

$$S(t) \to S_\infty = \varphi^{-1}(2R). \tag{23}$$

Let us now investigate the limiting regimes of motion of a thixotropic elastico-viscous medium. It is not difficult to show that, depending on the structural function $\varphi(s)$, the following cases may occur:

1) $\varphi'(0) < 1$ (this corresponds to $2R < S_\infty$): a) $\varphi''(s) = 0$, a stationary flow exists for any R; b) $\varphi''(s) > 0$, a stationary flow exists for $R < R^*$, where $2R^*$ is the solution of the equation $\varphi(x) = x$. For $R \geq R^*$ the body becomes quasi-solid.

2) $\varphi'(0) \geq 1$ (this corresponds to $2R \geq S_\infty$). A stationary flow does not exist even for very small R, and the liquid behaves as a quasi-solid body. This is realized for polymer melts which are characterized by the universal temperature-invariant function $N(s)$ of (8), for which $\varphi(S) \approx S$ at sufficiently small S.

In the general case, for an elastico-viscous thixotropic liquid, investigation of the dependence of the viscosity (in the case where a stationary flow exists) in the field of a longitudinal velocity gradient on the magnitude R requires determination of the structural functions $\varphi(s)$ and $N(s)$.

Results analogous to those considered above can be obtained in the investigation of axisymmetric and three-dimensional motion of an elastico-viscous liquid with a stagnation point. In the latter case there arises a certain nonuniqueness of the solution, connected with the relations between $\dot{\gamma}_{xx}$, $\dot{\gamma}_{yy}$, $\dot{\gamma}_{zz}$. Indeed, from the continuity equation $\dot{\gamma}_{xx} + \dot{\gamma}_{yy} + \dot{\gamma}_{zz} = 0$, with $\dot{\gamma}_{xx} = R$, the quantities $\dot{\gamma}_{yy}$ and $\dot{\gamma}_{zz}$ can be determined as follows: $\dot{\gamma}_{yy} = -c$, $\dot{\gamma}_{zz} = (c-1)R$, where c is an arbitrary number. This indeterminacy can be removed if it is assumed that, for sufficiently small t, the material behaves as an incompressible elastic body. In this case $c = \mu$, where $\mu = 1/2$ is Poisson' s ratio.

The results obtained make it possible to conclude that, at sufficiently high values of the longitudinal velocity gradient, stationary flow of an elastico-viscous liquid is impossible, and it will undergo quasi-brittle failure. In this case, the study of the conditions of failure of the liquid and the determination of the strength criterion acquire especially important significance.

The use of the rheological equations (1)–(3) makes it possible to solve a number of mechanical problems connected with the production of fibers and films.

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Note: Figure translations are in progress. See original paper for figures.