Yu. I. LYUBICH, V. A. TKACHENKO

1965

SovietRxiv

Abstract

Full Text

MATHEMATICS

Yu. I. LYUBICH, V. A. TKACHENKO

UNIQUENESS AND APPROXIMATION THEOREMS FOR THE LOCAL LAPLACE TRANSFORM

(Presented by Academician S. N. Bernstein on 16 II 1965)

In the note (1), in connection with the study of an abstract Cauchy problem, a certain generalization of the classical Laplace transform was proposed—the local Laplace transform (l.L.t.).* This generalization makes it possible to ignore such factors as the growth of the original as $t \to \infty$ or the boundedness of the interval of variation of t, which are very essential from the classical point of view.

In the present work we study the case where the l.L.t. is a meromorphic vector-function with values in an arbitrary Banach space B. In this situation the original may be associated with a formal series of residues, and then the problem of completeness (approximation) naturally arises. It turns out that completeness holds if the growth of the l.L.t. in the plane, measured in the sense of R. Nevanlinna ((4), pp.168–169), is bounded in an appropriate way.

1°. We begin by slightly extending, in comparison with (1), the definition of the l.L.t. This extension is closely connected with the definition of completely regular growth of a holomorphic function ((5), pp.182–183). It is dictated by certain applications.

Let M be some set of full (i.e., equal to 1) relative measure ((5), p.127) on the ray $\lambda \geq a \geq 0$. A locally integrable vector-function $f(\lambda)$ with values in B, defined at least on M, will be called a local Laplace transform (l.L.t.) on the interval $[0, T)$, where $0 < T \leq \infty$, if there exists a locally integrable vector-function $x(t)$ $(0 \leq t < T)$ such that for all $t \in [0, T)$ the representation

$$f(\lambda) = \int_0^t x(s) e^{-\lambda s}\, ds + \varepsilon(\lambda, t)$$

holds with a remainder term $\varepsilon(\lambda, t)$ satisfying the condition

$$\lim_{\lambda \to +\infty,\ \lambda \in M} \lambda^{-1} \ln \|\varepsilon(\lambda, t)\| \leq -t.$$

The linear class of functions which are l.L.t. on $[0, T)$ will be denoted by $\Lambda_T(B)$.

First of all, let us note that if a vector-function $f(\lambda) \in \Lambda_T(B)$ is holomorphic and of finite degree in some angle Γ_+ containing the ray $\lambda \geq a$, then the function $g[\varepsilon(\lambda, t)]$ will have the same property for any functional $g \in B^*$. From a theorem of V. Bernstein $((^6); (^5), pp.99–100)$ it follows that the indicator of the function $g[\varepsilon(\lambda, t)]$ on the positive ray does not exceed $-t$. Using now the Banach-Steinhaus theorem, it is easy to show that

$$\overline{\lim_{\lambda \to +\infty}} \lambda^{-1} \ln \|\varepsilon(\lambda, t)\| \leq -t.$$

* Variants of generalization close to the l.L.t. were indicated earlier in the works ([2],[3]).

Thus, $f(\lambda)$ proves to be an l.p.L. in the sense of (1). Therefore the following propositions are valid.

Lemma 1. *The original $x(t)$ of the function $f(\lambda) \in \Lambda_T(B)$ is determined uniquely (up to values on a set of measure zero).*

Lemma 2. *Let $f(\lambda) \in \Lambda_T(B)$ be a function holomorphic in T_t of finite degree. Put*

$$F(t) = \frac{1}{2\pi i} \int_a^\infty \frac{f(\lambda)}{\lambda} e^{\lambda t}\, d\lambda \qquad (\mathrm{Re}\, t < 0).$$

Then $F(t)$ is analytically continued into the domain

$$\Pi_T = \{\, t \mid \mathrm{Re}\, t < T,\ t \notin [0, T)\,\}$$

and

$$\lim_{s \downarrow 0} [F(t + is) - F(t - is)] = \int_0^t x(s)\, ds \qquad (0 < t < T).$$

2°. As an illustrative example, consider the integral equation

$$\int_0^1 a(s)\xi(s + t)\, ds = 0 \qquad (0 < t < T;\ T \leq \infty)$$

with kernel $a(s) \in \mathcal{L}(0, 1)$. Put

$$A(\lambda) = \int_0^1 a(s) e^{\lambda s}\, ds, \qquad \varphi(s, \lambda) = \frac{e^{\lambda s}}{A(\lambda)} \int_0^1 a(u) e^{\lambda u}\, du \int_u^s \xi(v) e^{-\lambda v}\, dv.$$

We shall treat the functions $\xi(s + t)$, $\varphi(s, \lambda)$ as vector-functions $x(t)$, $f(\lambda)$ with values in $\mathcal{L}(0, 1)$. The "characteristic" function $A(\lambda)$ has quite regular growth ((5), p. 324). Using this fact, it is easy to show that $f(\lambda) \in \Lambda_T(\mathcal{L}(0, 1))$ and that $-x(t)$ is the original. Hence, on the basis of Lemma 1, it follows that the solution $\xi(s)$ $(0 < s < T + 1)$ is uniquely determined by its values on $(0, 1)$. This uniqueness theorem was proved earlier in (7) *.

3°. Consider the case of an entire $f(\lambda) \in \Lambda_T(B)$. Put

$$M_f(r) = \max_{|\lambda| \le r} \|f(\lambda)\|, \qquad \sigma_f = \lim_{r\to\infty} r^{-1} M_f(r)$$

(σ_f is the "degree" of the function $f(\lambda)$). We shall assume $\sigma_f < \infty$ and introduce the growth indicator

$$h_f(\theta) = \overline{\lim_{r\to\infty}}\, r^{-1} \ln \|f(re^{i\theta})\|.$$

The following analogue of the Wiener–Paley theorem holds.

Theorem 1. *Let $f(\lambda) \in \Lambda_T(B)$ be an entire function of degree $\sigma_f < T$ and*

$$h = \max(0, h_f(-\pi)).$$

Then

$$f(\lambda) = \int_0^h x(s) e^{-\lambda s}\, ds.$$

Proof. For every $g \in B^*$ we have

$$\Phi_g(t) \equiv \frac{d}{dt} g[F(t)] = g[F'(t)] = \frac{1}{2\pi i} \int_a^\infty g[f(\lambda)] e^{\lambda t}\, d\lambda \qquad (\mathrm{Re}\, t < 0).$$

Thus, the function $\Phi_g(-t)$ is (up to an integral summand) the Borel associate of the function $g[f(\lambda)]$. Its singularities, by Lemma 2 and the inequality $\sigma_f < T$, can lie only on the negative real half-axis. But then, by the well-known theorem of Pólya, the indicator diagram of the function $g[f(\lambda)]$ is contained in the segment $[-h, 0]$, and the singularities of the function $\Phi_g(-t)$ also lie there. Therefore $F'(t)$ is analytically continued through (h, ∞), and hence

$$\lim_{s\downarrow 0} [F(t + is) - F(t - is)] = \mathrm{const} \qquad (t > h).$$

* In (7) the well-known theorem of Titchmarsh on convolution is used ((8), pp. 445–446).

It can also be easily proved with the aid of l.p.L.

By Lemma 2, $x(t) = 0$ almost everywhere in (h, T). Now

$$f(\lambda) = \int_0^h x(s) e^{-\lambda s}\, ds + \varepsilon(\lambda, t) \qquad (h < t < T).$$

The function $\varepsilon(\lambda, t)$ turns out to be independent of t and to be an entire function of λ of finite degree. The indicator diagram for $g[\varepsilon(\lambda, t)]$ is contained in $[-h, 0]$. But the indicator of $g[\varepsilon(\lambda, t)]$ along the ray $\lambda > 0$ does not exceed $-t < -h$. Therefore $g[\varepsilon(\lambda, t)] = 0$, whence $\varepsilon(\lambda, t) = 0$ for $t > h$.

Remark. Theorem 1 is inverted in an obvious way.

In the corollaries to Theorem 1 given below its conditions are assumed to be satisfied.

Corollary 1. Almost everywhere in (h, T), $x(t) = 0$.

Corollary 2. If $h_f(-\pi) \le 0$, then $f(\lambda) = 0$, and $x(t) = 0$ almost everywhere in $(0, T)$.

4°. Let now $f(\lambda)$ be a meromorphic function. Introduce the Nevanlinna growth function

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \|f(re^{i\theta})\| \, d\theta.$$

In this case the analogue of degree is $\tau_f = \overline{\lim}_{r\to\infty} r^{-1} m_f(r)$. Put

$$\tau_f^- = \inf_M \left\{ \overline{\lim_{r\to\infty,\ r\in M}} \frac{1}{\pi r} \int_{\pi/2}^{3\pi/2} \ln^+ \|f(re^{i\theta})\| \, d\theta \right\},$$

where M ranges over the class of all sets of full relative measure on the ray $r > 0$.

Lemma 3. Let $\varphi(\lambda)$ be a scalar entire function of finite degree and

$$\omega = \overline{\lim_{r\to\infty,\ r\in M}} \frac{1}{\pi r} \int_{\pi/2}^{3\pi/2} \ln^+ |\varphi(re^{i\theta})| \, d\theta,$$

where M is a set of full relative measure. Then for any δ $(0 < \delta < \pi/2)$ there exists a constant c_δ, depending only on δ, such that

$$h_\varphi(\theta) \le c_\delta \omega \qquad (\pi/2 + \delta \le \theta \le 3\pi/2 - \delta).$$

We shall regard the constant c_δ as the best one. Introduce the absolute constant*

$$c = \inf_\delta c_\delta = \lim_{\delta\to\pi/2} c_\delta.$$

Theorem 2. Let $f(\lambda) \in \Lambda_T(B)$ be a meromorphic function of finite degree with set of poles $\{\lambda_k\}_1^\infty$, and let $x(s)$ $(0 \le s < T)$ be its original. On any segment $(c\tau_f^-, T_1)$ $(\tau_f^- < T_1 < T)$ there exists a sequence of aggregates converging in the mean to $x(s)$,

$$x_N(s) = \sum_{k=1}^{N} \sum_{m=0}^{n_k-1} \alpha_{m,k}^{(N)} e^{\lambda_k s} P_{m,k}(s),$$

where $P_{m,k}(s)$ are polynomials determined by the equality

$$P_{m,k}(s) = e^{-\lambda_k s} \left(\frac{d}{ds} - \lambda_k \right)^m \operatorname{Res} \left[f(\lambda) e^{\lambda s} \right]_{\lambda=\lambda_k},$$

n_k is the multiplicity of the pole λ_k, and $\alpha_{m,k}^{(N)}$ are scalar coefficients.

If the original $x(s)$ is continuous, then uniform approximation by aggregates of the same form can be carried out.

* It can be shown that $\pi/2 \le c \le 8/3$.

Let us outline the proof, choosing, for definiteness, approximation in the mean.

Consider the space of vector-functions $\mathcal{L} = \mathcal{L}(B; (0, T_0))$, where $T_0 = T_1 - c\tau,\ \tau = \tau_f^-$. The representation

$$e^{\lambda s} \varepsilon(\lambda, s) = e^{\lambda s} \left(f(\lambda) - \int_0^s x(u) e^{-\lambda u}\, du \right) = \int_0^v x(t+s) e^{-\lambda t}\, dt + e^{\lambda s} \varepsilon(\lambda, v+s)$$

shows that $\hat{f}(\lambda) = e^{\lambda s} \varepsilon(\lambda, s)$, as an element of $\mathcal{L}$, is an L. P. L. on $[0, T - T_0)$ with original $\hat{x}(t) = x(t+s)$. Suppose $\hat{g} \in \mathcal{L}^*$ annihilates the closed linear hull of the vector-functions $e^{\lambda_k s} P_{m,k}(s)$. We shall show that $\hat{g}[\hat{x}(t)] = 0$ if $c\tau \le t < T - T_0$. Note that $\varphi(\lambda) = \hat{g}[\hat{f}(\lambda)]$ is an entire function of finite degree from $\Lambda_{T-T_0}(B^1)$. By the definition of the L. P. L. $h_\varphi(0) \le 0$, and by Lemma 3 $h_\varphi(\theta) \le c\delta\tau (\pi/2 + \delta \le \theta \le 3\pi/2 - \delta)$. By the Phragmén-Lindelöf principle, the degree of the function $\varphi(\lambda)$ does not exceed $c\tau$ (in view of the arbitrariness of δ). According to Corollary 1 of Theorem 1, the original $\hat{g}[\hat{x}(t)]$ is equal to zero on $[c\tau, T - T_0)$ everywhere (since it is continuous). In particular, $\hat{g}[\hat{x}(c\tau)] = 0$. Taking into account the choice of the functional $\hat{g}$, by the Hahn-Banach theorem we conclude that the element $x(c\tau) = x(s + c\tau)$ $(0 \le s \le T_1 - c\tau)$ is approximable in $\mathcal{L}$ by linear combinations of functions of the form $e^{\lambda_k s} P_{m,k}(s)$.

5°. The growth restriction on $f(\lambda)$ in Theorem 2 cannot be substantially weakened. This is shown by the following theorems, close to the uniqueness theorems from ([9,10]).

Theorem 3. Let the function $\rho(r) > 0$ $(r > 0)$ be such that $r^{-1}\rho(r) \to \infty$ $(r \to \infty)$. Then there exist a Hilbert space H and an entire function $f(\lambda) \in \Lambda_\infty(H)$ with original not identically zero such that $m_f(r) \le \rho(r)$ $(r > 0)$.

Theorem 4. For any $\tau > 0$ there exist a Hilbert space H and an entire function of finite degree $f(\lambda) \in \Lambda_\infty(H)$ for which $\tau_f^- = \tau$ and whose original is nonzero everywhere on the interval $[0, c_1\tau]$, where $c_1 > 0$ is an absolute constant.

6°. Let us apply Theorem 2 to the situation considered in 2°. Here the L. P. L. $f(\lambda)$ is a meromorphic function of finite degree. From the completely regular growth of the function $A(\lambda)$ it follows that $\tau_f^- = 0$. By Theorem 2, every solution $\xi(s)$ $(0 < s < T + 1)$ can be approximated with arbitrary accuracy in the mean on each interval $(0, T_1 + 1)$ for $0 < T_1 < T$ by linear combinations of solutions of the form $s^j e^{\lambda s}$. This result was essentially established by other methods in ([11,12]).

In conclusion we note that from Theorem 2 one can obtain an approximation theorem for solutions of an abstract Cauchy problem by exponential solutions.

Kharkov State University
named after A. M. Gorky

Kharkov Polytechnic Institute
named after V. I. Lenin

Received
20 I 1965

REFERENCES

1. Yu. I. Lyubich, DAN, **154**, No. 1 (1964).
2. V. A. Ditkin, DAN, **123**, No. 3 (1958).
3. L. Berg, Studia Math., **21**, 215 (1962).
4. R. Nevanlinna, *Univalent Analytic Functions*, Moscow–Leningrad, 1941.
5. B. Ya. Levin, *Distribution of Roots of Entire Functions*, Moscow, 1956.
6. V. Bernstein, Ann. Scuola norm. sup. Pisa (2), **2**, 381 (1933).
7. Yu. I. Lyubich, Matem. sbornik, **38** (80), No. 2 (1956).
8. E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Moscow –Leningrad, 1948.
9. Yu. I. Lyubich, DAN, **130**, No. 5 (1960).
10. Yu. I. Lyubich, UMN, **16**, No. 5 (1961).

11. L. Schwartz, Ann. Math., **48**, 857 (1947).

12. J.-P. Kahane, Ann. inst. Fourier, **5**, 39 (1955).

Note: Figure translations are in progress. See original paper for figures.