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Abstract

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ON ESTIMATES OF THE GROWTH OF SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS IN PARTIAL DERIVATIVES OF HYPERBOLIC TYPE

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In notes ($^{1-5}$), stability criteria (boundedness on a half-axis) of solutions were obtained for a broad class of ordinary differential-difference equations. In the present note we have succeeded in obtaining a necessary and sufficient criterion for the growth of solutions for differential-difference equations of the form

$$\frac{\partial^{p_1+p_2+\cdots+p_n}Y}{\partial t_1^{p_1}\partial t_2^{p_2}\cdots\partial t_n^{p_n}} - A(t_1,t_2,\dots,t_n)Y(t_1-a_1,t_2-a_2,\dots,t_n-a_n) =$$

$$= f(t_1,t_2,\dots,t_n),$$

$$p_1,p_2,\dots,p_n \geqslant 1; \qquad a_k = a_k(t_1,t_2,\dots,t_n) \geqslant 0 \quad (0 \leqslant t_k < \infty),$$

where $A(t_1,t_2,\dots,t_n)$ is a linear operator-function acting in a complex Banach space $\mathfrak{E}$, and $f(t_1,t_2,\dots,t_n)$, $Y(t_1,t_2,\dots,t_n)$ are continuous vector-functions with range belonging to $\mathfrak{E}$. The main result may be formulated in the form of the following theorem.

Theorem. *Consider the boundary-value problem*

$$-\frac{\partial^{p_1+p_2+\cdots+p_n}Y}{\partial t_1^{p_1}\partial t_2^{p_2}\cdots\partial t_n^{p_n}} - A(t_1,t_2,\dots,t_n)Y(t_1-a_1,t_2-a_2,\dots,t_n-a_n) =$$

$$= f(t_1,t_2,\dots,t_n),$$

$$p_1,p_2,\dots,p_n \geqslant 1; \qquad a_k = a_k(t_1,t_2,\dots,t_n) \geqslant 0$$

$$(0 \leqslant t_1,t_2,\dots,t_n < \infty),$$

$$\frac{\partial^{q_k}Y}{\partial t_k^{q_k}} = \varphi_{q_k}(t_1,t_2,\dots,t_n)$$

$$(t_k \leqslant 0;\ q_k = 0, 1, \dots, p_k - 1;\ k = 1, 2, \dots, n). \tag{1}$$

Let the continuous operator-function $A(t_1, t_2, \dots, t_n)$ *satisfy the conditions:*

1) *the family of operators is compact;*
2) $A(t_1, t_2, \dots, t_n)$ *has weak variation at infinity, i.e., for any* $\varepsilon > 0$ *there exists a* $T = T(\varepsilon) > 0$ *such that for all* t_k', $t_k'' > T$ *and* $|t_k' - t_k''| < 1$,

$$\|A(t_1', t_2', \dots, t_n') - A(t_1'', t_2'', \dots, t_n'')\| < \varepsilon.$$

The functions $a_k(t_1, t_2, \dots, t_n) \geqslant 0$ *will be assumed continuous, bounded, and satisfying condition 2).*

Consider all possible limit operators A_ω and the corresponding limit values of the functions $a_k^{(\omega)}$, to which the families of operators and functions converge on common sequences:

$$A(t_1^{(m)},\, t_2^{(m)},\, \dots,\, t_n^{(m)}) \to A_\omega; \qquad a_k(t_1^{(m)},\, t_2^{(m)},\, \dots,\, t_n^{(m)}) \to a_k^{(\omega)}$$

$$(t_k^{(m)} \to \infty).$$

Then, in order that the solution Y of the boundary-value problem (1) satisfy the condition $\|Y\| \leq C \exp[\alpha(t_1 + t_2 + \dots + t_n)]$ for all $\|f\| \leq C \exp[\alpha(t_1 + t_2 + \dots + t_n)]$, it is necessary and sufficient that, for all $\Lambda_\omega \in \operatorname{sp.} A_\omega$ and the corresponding $a_k^{(\omega)}$, all roots $z(z_1, z_2, \dots, z_n)$ of the equation

$$z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} - \Lambda_\omega \exp\left[-\sum_{k=1}^{n} a_k^{(\omega)} z_k\right] = 0 \tag{2}$$

satisfy the condition

$$\sup_{(z)} \left(\min_k \operatorname{Re} z_k\right) < \alpha, \qquad k = 1, 2, \dots, n. \tag{3}$$

Proof of the theorem will be carried out according to the following plan:

I. First we prove the theorem for the case when $A(t_1, t_2, \dots, t_n) \equiv A$; $a_k(t_1, t_2, \dots, t_n) \equiv a_k$ are constants. By replacing the unknown function, we reduce the boundary-value problem (1) to a problem with zero initial functions: $\varphi_{q_k}(t_1, t_2, \dots, t_n) \equiv 0$, which is equivalent to the operator equation

$$Y - A S_1^{p_1} S_2^{p_2} \cdots S_n^{p_n} K_1 K_2 \cdots K_{nY} = S_1^{p_1} S_2^{p_2} \cdots S_n^{p_n} f, \tag{4}$$

where

$$S_i f = \int_0^{t_i} f(t_1, \dots, \tau, \dots, t_n)\, d\tau$$

is the integration operator, and

$$K_i f = f(t_1, \dots, t_i - a_i, \dots, t_n)$$

is the "delay" operator. It is convenient to consider equation (4) in the space of continuous functions $\widetilde{E}_0$ such that $f(t_1, \dots, t_i, \dots, t_n) = 0$ $(t_i < 0)$. In the space $\widetilde{E}_0$ we approximate the operators K_i and S_i by the operators

$$K_i(m) f = f(t_1, \dots, t_i - m_i/m, \dots, t_n),$$

$$\sigma_i(m) f = \frac{1}{m} \sum_{k=0}^{\infty} f(t_1, \dots, t_i - k/m, \dots, t_n), \qquad m_i/m \to a_i$$

$$(m_i, m \to \infty;\ m_i, m \text{ are natural numbers}).$$

We approximate equation (4) by the operator-difference equation:

$$Y_m - A \prod_{i=1}^{n} \sigma_i^{p_i}(m) K_i(m) Y_m = \prod_{i=1}^{n} \sigma_i^{p_i}(m) f. \tag{5}$$

It is easy to show that $\|Y_m - Y\| < \varepsilon$ for $m \geq N$, uniformly on every finite interval.

Consider the auxiliary function

$$\Phi_m = \Phi_m(z_1, z_2, \dots, z_n) = \sum_{k_1, k_2, \dots, k_n = 0}^{\infty} Y_m\left(\frac{k_1}{m}, \frac{k_2}{m}, \dots, \frac{k_n}{m}\right) z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}.$$

Apply the operators $K_i(m)$ and $\sigma_i(m)$ to Φ_m, taking into account that $Y_m(t_1, t_2, \dots, t_n)$ belongs to $\widetilde{E}_0$.

We shall have:

$$K_i(m)\Phi_m = z_i^{m_i} \Phi_m; \qquad \sigma_i(m)\Phi_m = \frac{1}{m(1 - z_i)} \Phi_m.$$

The function Φ_m obviously satisfies equation (5) if, in place of f, we substitute the function

$$F_m(z_1, z_2, \dots, z_n) = \sum_{k_1, k_2, \dots, k_n = 0}^{\infty} f\left(\frac{k_1}{m}, \frac{k_2}{m}, \dots, \frac{k_n}{m}\right) z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}.$$

From equation (5) we find the unknown function Φ_m and obtain the formula

$$\Phi_m(z_1, z_2, \dots, z_n) = \left(\left(\frac{1-z_1}{1/m}\right)^{p_1} \left(\frac{1-z_2}{1/m}\right)^{p_2} \cdots \left(\frac{1-z_n}{1/m}\right)^{p_n} - A z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}\right)^{-1} F_m(z_1, z_2, \dots, z_n). \tag{6}$$

The proof of the theorem is based on formula (6). The growth of the solution $Y_m(t_1, t_2, \dots, t_n)$ depends on the position of the singular points of the operator

$$B_m(z_1, z_2, \dots, z_n) = \left(\left(\frac{1-z_1}{1/m}\right)^{p_1} \left(\frac{1-z_2}{1/m}\right)^{p_2} \cdots \left(\frac{1-z_n}{1/m}\right)^{p_n} - A z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}\right)^{-1} \tag{7}$$

relative to the points $z_k = 0$. If we put $z_k = \xi_k^{1/m}$, then

$$B_m(\xi_1, \xi_2, \dots, \xi_n) \to B(\xi_1, \xi_2, \dots, \xi_n) \qquad (m \to \infty),$$

where

$$B(\xi_1, \xi_2, \dots, \xi_n) = \left((-\ln \xi_1)^{p_1} (-\ln \xi_2)^{p_2} \cdots (-\ln \xi_n)^{p_n} - A \xi_1^{a_1} \xi_2^{a_2} \cdots \xi_n^{a_n}\right)^{-1}.$$

It is convenient to replace $-\ln \xi_k = z_k$, $\xi_k = e^{-z_k}$. We shall have

$$B_1(z_1, z_2, \dots, z_n) = \left(z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} - A \exp\left[-\sum_{k=1}^{n} a_{kz} k\right]\right)^{-1}. \tag{8}$$

Suppose that for all $\lambda \in \operatorname{sp.} A$ all roots of the equation

$$z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} - \lambda \exp\left[-\sum_{k=1}^{n} a_{kz} k\right] = 0 \tag{2'}$$

satisfy condition (3). Then the operator (8) will be holomorphic for all z_k with $\operatorname{Re} z_k \geq \alpha_1$ $(\alpha_1 < \alpha)$, and the operators (7), obviously, are holomorphic for all $|z_k| \leq \exp[-\alpha_1/m]$, beginning with $m \geq N$. This makes it possible to estimate the Taylor coefficients $b_{k_1 k_2 \dots k_n}$ of the function $B_m(z_1, z_2, \dots, z_n)$, and consequently also the function $\Phi_m(z_1, z_2, \dots, z_n)$ from formula (6), on the basis of Cauchy' s inequality in strengthened form,

$$\|b_{k_1 k_2 \dots k_n}\| \leq \exp\left[\frac{\alpha_1}{m}(k_1 + k_2 + \dots + k_n)\right] \times$$

$$\times \oint \oint \cdots \oint_{|z_k| = e^{-\alpha_1/m}} \|B_m(z_1, z_2, \dots, z_n)\| \, |dz_1| \, |dz_2| \cdots |dz_n| \qquad (\alpha_1 < \alpha).$$

Carrying out sufficiently precise estimates, we shall have:

$$\|Y_m(k_1/m, k_2/m, \dots, k_n/m)\| \leq C \exp[\alpha(k_1/m + k_2/m + \dots + k_n/m)]$$

$$(m \geq N);$$

C does not depend on m. Hence follows the estimate for $Y(t_1, t_2, \dots, t_n)$

$$\|Y(t_1, t_2, \dots, t_n)\| \leq C \exp[\alpha(t_1 + t_2 + \dots + t_n)] \qquad (\alpha > 0).$$

The sufficiency of the theorem is proved.

II. Suppose that, for some $\lambda_0 \in \text{sp.}\, A$, the roots $z(z_1, z_2, \dots, z_n)$ of equation $(2')$ satisfy the condition

$$\sup_{(z)} \left(\min_k \operatorname{Re} z_k\right) = \alpha_0 \geq \alpha.$$

Consider two

cases: $a_0 > a$ and $a_0 = a$. In the case $a_0 > a$ there exist roots z_k with $\operatorname{Re} z_k \geq a_1 > a$. One can choose such a $\lambda_0 \in \text{sp.}\, A$ and such roots z_k with $\operatorname{Re} z_k \geq a_1$ for which there exists a sequence of regular points λ_m of the operator A, converging to $\lambda_0 \in \text{sp.}\, A$ ($\lambda_m \to \lambda_0$, $m \to \infty$). Then one can choose sequences of vectors f_m and e_m such that

$$Af_m = \lambda_m f_m + e_m; \quad \|f_m\| = 1; \quad \|e_m\| \to 0.$$

If in formula (6) we put $f(t_1, t_2, \dots, t_n) = f_m$ and estimate from below the Taylor coefficients $Y_m(k_1/m, k_2/m, \dots, k_n/m)$, then one can obtain an estimate for $Y(t_1, t_2, \dots, t_n)$:

$$\|Y(t_1, t_2, \dots, t_n)\| \geq C \exp\left[a_1(t_1 + t_2 + \dots + t_n)\right], \quad C > 0 \ (a_1 > a).$$

In this case the necessity of the theorem is proved.

In the case when

$$\sup_{(z)} \left(\min_k \operatorname{Re} z_k \right) = a_0 = a,$$

one can perturb the operator A by an arbitrarily small amount in norm so that, for some $\lambda_0 \in \operatorname{sp.} A$, $a_0 > a$, and obtain the preceding case. The theorem is proved.

III. In the case when the operator-function $A(t_1, t_2, \dots, t_n)$ and the functions $a_k(t_1, t_2, \dots, t_n)$ are not constant, the proof of the theorem relies on formula (6) and conditions 1), 2) of the theorem.

Remark 1. It should be noted that the boundary-value problem (1) for bounded functions $f(t_1, t_2, \dots, t_n)$ always has an unbounded solution. This follows from the fact that the roots of equation (2), for $n > 1$, always satisfy the condition $\sup_{(z)}(\min_k \operatorname{Re} z_k) \geq 0$, whereas for boundedness it is necessary that $\sup_{(z)}(\min_k \operatorname{Re} z_k) < 0$.

Remark 2. For $n = 1$ and $a = 0$ one obtains the necessary and sufficient boundedness criterion for the solution that was obtained by us in note [1].

Remark 3. In the case when all $a_k(t_1, t_2, \dots, t_n) \equiv 0$, the growth criterion can be formulated as follows: in order that the solution of equation (1) satisfy the inequality $\|Y(t_1, t_2, \dots, t_n)\| \leq C \exp[\alpha(t_1 + t_2 + \dots + t_n)]$ for all $\|f\| \leq C \exp[\alpha(t_1 + t_2 + \dots + t_n)]$, it is necessary and sufficient that, for all limiting operators A_ω, the condition hold:

$$\lambda_\omega \in \operatorname{sp.} A_\omega; \quad \max \operatorname{Re} \sqrt[p]{\lambda_\omega} < \alpha \quad (p = p_1 + p_2 + \dots + p_n). \tag{9}$$

Condition (9), as is easy to see, is equivalent to the fact that all roots $z(z_1, z_2, \dots, z_n)$ of the equation $z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} - \lambda_\omega = 0$ satisfy condition (3). Criterion (9) was obtained by M. A. Rutman in note [6].

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Note: Figure translations are in progress. See original paper for figures.

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