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Abstract

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MATHEMATICS

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ON LIMIT RELATIONS AND ASYMPTOTIC FORMULAS FOR POLYNOMIALS ORTHOGONAL ON THE UNIT CIRCLE

(Presented by Academician S. N. Bernstein on 4 VIII 1964)

1. Let

$$P_n(z) = \varkappa_n z^n + \ldots; \qquad \varkappa_n > 0, \quad n = 0, 1, 2, \ldots,$$

be polynomials orthonormal on the unit circle with respect to the measure $d\sigma(\theta)$, i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} P_n(e^{i\theta})\,\overline{P_m(e^{i\theta})}\,d\sigma(\theta) = \delta_{nm},$$

where $\sigma(\theta)$ is a bounded nondecreasing function with an infinite set of points of increase. Let

$$\int_0^{2\pi} \ln p(\theta)\,d\theta > -\infty, \tag{1}$$

where $p(\theta)$ is the derivative $\sigma'(\theta)$, which exists almost everywhere. As is known, in this case the function

$$\pi(z, p) = \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi} \ln p(\theta)\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right\}$$

is regular in $|z| < 1$, and

$$\lim_{n\to\infty} P_n^*(z) = \pi(z,p) \qquad \left(P_n^*(z) = z^n \overline{P_n\left(\frac{1}{z}\right)}\right),$$

$$\lim_{n\to\infty} \varkappa_n = \varkappa = \pi(0,p), \tag{2}$$

where the limit relation (2) holds uniformly for $|z| \leq r < 1$. S. N. Bernstein (^{1}) and G. Szegő (^{2}) were the first to pose and solve the problem of the uniform limit relation

$$\lim_{n\to\infty} P_n^*(e^{i\theta}) = \pi(e^{i\theta},p), \qquad \theta \in [0,2\pi], \tag{3}$$

where

$$\pi(e^{i\theta}) = \lim_{r\to 1-0} \pi(re^{i\theta}),$$

and found an estimate for the difference

$$\rho_n(\theta) = |P_n^*(e^{i\theta}) - \pi(e^{i\theta})|$$

in the case when $\sigma(\theta)$ is an absolutely continuous function: $\sigma(\theta) \in aC^*$, and the weight $p(\theta)$ is a positive 2π-periodic continuous function satisfying a Dini-Lipschitz condition of order > 1.

* $d\sigma(\theta) = p(\theta)\, d\theta$, $\quad p(\theta)$ is a nonnegative summable function.

Ya. L. Geronimus (^{3}) generalized the formulation of this problem, extending it also to the case of an interior interval $[\alpha,\beta] \subset [0,2\pi]$. In the present note new cases are given for the existence of the limiting relation (3) at each point $\theta \in [\alpha,\beta]$, almost everywhere on $[0,2\pi]$ or on $[\alpha,\beta]$, uniformly on $[0,2\pi]$ or on $[\alpha,\beta]$.

We denote, as usual, the integral modulus of continuity of a 2π-periodic function $f(\theta) \in \mathcal{L}_q(0,2\pi)$, $1 \leq q < \infty$, by

$$\Omega_q(\delta,f) = \sup_{|h|\leq\delta} \|f(\theta+h) - f(\theta)\|_{\mathcal{L}_q(0,2\pi)},$$

and the modulus of continuity of a 2π-periodic continuous function $g(\theta) \in C(0,2\pi)$ by

$$\Omega(\delta,g) = \max_{|h|\leq\delta} \|g(\theta+h) - g(\theta)\|_{C(0,2\pi)}.$$

If $f(\theta) \in \mathcal{L}_q(\alpha, \beta)$, $g(\theta) \in C(\alpha, \beta)$, then

$$\omega_q(\delta, f) = \sup_{|h| \leq \delta \leq \delta_0} \|f(\theta + h) - f(\theta)\|_{\mathcal{L}_q(\alpha', \beta')},$$

$$\omega(\delta, g) = \max_{|h| \leq \delta \leq \delta_0} \|g(\theta + h) - g(\theta)\|_{C(\alpha', \beta')},$$

$$[\alpha', \beta'] \subset [\alpha, \beta], \qquad \delta_0 = \min(\alpha' - \alpha, \beta - \beta').$$

2. Theorem 1. Suppose that on $[0, 2\pi]$

$$\sigma(\theta) \in aC,$$

$$0 < B_0 \leq p(\theta) \in C(0, 2\pi); \tag{4}$$

$$\frac{\Omega(t, p)}{t} \in \mathcal{L}_1. \tag{5}$$

Then, starting from some $n \geq N_0$, we have

$$\rho_n(\theta) \leq B_1 \int_0^{\delta_n} \frac{\Omega(t, p)}{t}\, dt + B_2 \frac{1}{\delta_n} \Omega\left(\frac{1}{n}, p\right)^*, \tag{6}$$

where δ_n is chosen so that

$$\frac{1}{\delta_n} \Omega\left(\frac{1}{n}, p\right) = o(1) \quad \text{as } n \to \infty.$$

If, instead of conditions (4) and (5), the conditions

$$p(\theta) \leq B_2, \qquad \frac{1}{p(\theta)} \in \mathcal{L}_2, \tag{7}$$

and, on $[\alpha, \beta]$,

$$0 < B_0 \leq p(\theta) \in C(\alpha, \beta); \tag{8}$$

$$\frac{\omega(t, p)}{t} \in \mathcal{L}_1, \tag{9}$$

hold, then, starting from some $n \geq N_1$, we have for $\theta \in [\alpha', \beta']$

$$\rho_n(\theta) \leq B_3 \int_0^{\varepsilon_n} \frac{\omega(t,p)}{t}\, dt + \frac{B_4}{\varepsilon_n}\Omega_2\left(\frac{1}{n}, \frac{1}{p}\right) + B_5\Omega_1^{1/2}\left(\frac{1}{n}, \frac{1}{p}\right), \tag{10}$$

and ε_n is chosen so that

$$\frac{1}{\varepsilon_n}\Omega^2\left(\frac{1}{n}, \frac{1}{p}\right) = o(1) \quad \text{as } n \to \infty.$$

If $\Omega(\delta, p) = O\left\{\left(\ln \frac{1}{\delta}\right)^{-(1+\varepsilon)}\right\}$, $\varepsilon > 0$, then, putting in (6) $\delta_n = (\ln n)^{-1}$, we obtain the estimate of S. N. Bernstein and G. Szegő.

3. Theorem 2. Suppose that condition (1) is satisfied, and on $[\alpha, \beta]$

$$\sigma(\theta) \in aC, \qquad 0 < B_0 \leq p(\theta) \in C(\alpha, \beta); \tag{11}$$

$$\frac{\omega(t,p)}{t} \in \mathcal{L}_2; \tag{12}$$

$$(Z_1): \qquad \delta \int_\delta^{\delta_0} \frac{\omega(t,p)}{t^2}\, dt = O\{\omega(\delta, p)\}^*. \tag{13}$$

Then, uniformly for $\theta \in [\alpha', \beta']$, (3) holds.

We note that conditions (12) and (13) are satisfied if

$$\omega(\delta, p) = O\left\{\delta^\nu \left(\ln \frac{B}{\delta_0}\right)^{-\mu}\right\} \tag{14}$$

with

$$0 \leq \mu < (1-\nu)\ln\frac{B}{\delta_0} - 1, \qquad B > \delta_0, \qquad \tfrac{1}{2} \leq \nu < 1.$$

Theorem 3. Let $\rho^{-1}(\theta) \in \mathcal{L}_1$, and suppose that on $[\alpha, \beta]$ condition (9), condition (Z_1) (or the equivalent conditions), and, instead of condition (12), the condition

$$(Z) \qquad \int_0^\delta \frac{\omega(t,p)}{t}\, dt = O\{\omega(\delta, p)\}^{**} \tag{15}$$

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[*] $B_0, B_1, B_2, \ldots$ are various constants.

hold. Then, uniformly for $\theta \in [\alpha', \beta']$, (3) holds.

We note that conditions (Z) and (Z_1) are satisfied if $\omega(\delta, p)$ satisfies inequality (14) with

$$0 \le \mu < (1-\nu)\ln\frac{B}{\delta_0} - 1, \quad B > \delta_0, \quad 0 < \nu < 1.$$

Remark 1. Under the hypotheses of Theorem 3 we have, uniformly for $\theta \in [\alpha', \beta']$ and for all $n \ge N_2$, the lower estimate

$$\rho_n(\theta) \ge B_4 \omega\left(\frac{1}{n}\right).$$

If

$$B_5 \delta^\alpha \le \omega(\delta, p) \le B_6 \delta^\beta, \qquad 0 \le \alpha < \beta < 1,$$

then

$$\rho_n(\theta) \ge B_7 n^{-\beta(1-\alpha)/(1-\beta)}.$$

Theorem 4. Let $\rho^{-1}(\theta) \in \mathcal{L}_1$, and suppose that on $[\alpha, \beta]$ condition (9) holds and

$$\frac{\omega(t, p)}{t} \ln\frac{1}{t} \in \mathcal{L}_1.$$

Then, uniformly for $\theta \in [\alpha', \beta']$, (3) holds.

We prove Theorems 2-4 by the method of Ya. L. Geronimus (3), applying a local analogue of a theorem of A. Zygmund, which makes it possible, in terms of the modulus of continuity of a given function $f(\theta)$, to estimate from above the modulus of continuity of the conjugate function $\tilde{f}(\theta)$ (5):

$$\omega(\delta, \tilde{f}) = O\left\{\int_0^\delta \frac{\omega(t, f)}{t}\,dt + \delta\int_\delta^{\delta_0} \frac{\omega(t, f)}{t^2}\,dt\right\} \tag{16}$$

under the condition that

$$f(\theta) \in \mathcal{L}_1, \qquad f(\theta) \in C(\alpha, \beta), \qquad \frac{\omega(t, f)}{t} \in \mathcal{L}_1.$$

Remark 2. Let

$$B_9 \delta^\nu \lambda(\delta) \le \omega(\delta, p) \le B_8 \delta^\nu \lambda(\delta), \tag{17}$$

where $0 < \nu < 1$; $\lambda(\sigma)$ is an almost nondecreasing function on $[0, \delta_0]$, i.e.

$$\lambda(\delta_2) \le B_{10}\lambda(\delta_1) \qquad (\delta_2 > \delta_1,\ B_{10} \ge 1). \tag{18}$$

Starting from inequality (16), we obtain

$$\omega(\delta, \pi) = O\left\{\int_0^\delta \frac{\omega(t,p)}{t}\,dt\right\},$$

and in Theorem 2, instead of condition (Z_1), one may take condition (17).

* Condition (Z_1) is equivalent to conditions $(S_1), (L_1), (P_1), (B_1)$ in the notation of (4).

** Condition (Z) is equivalent to conditions $(S), (L), (P), (B)$ in the notation of (4).

4. **Theorem 5.** Suppose that on $[0, 2\pi]$ $\sigma(\theta) \in aCu$,

$$0 < B_0 \le p(\theta) \le B_1, \qquad \Omega_2^2(t,p)\frac{1}{t}\ln\frac{1}{t} \in \mathcal{L}_1 \tag{19}$$

or

$$p(\theta) \in \mathcal{L}_q, \qquad \frac{1}{p(\theta)} \in \mathcal{L}_{q'}, \qquad \frac{1}{q} + \frac{1}{q'} = 1, \qquad 1 < q \le 2, \tag{20}$$

$$\Omega_q(t,p)\frac{1}{t}\ln\frac{1}{t} \in \mathcal{L}_1.$$

Then for almost all $\theta \in [0, 2\pi]$, (3) holds.

The proof is based on a theorem of G. Rademacher and D. E. Menshov [6] and on estimates of Ya. L. Geronimus ([3], Table I).

Theorem 6. Suppose that on $[0, 2\pi]$ $\sigma(\theta) \in aC$, $\dfrac{1}{p(\theta)} \in \mathcal{L}_2$, and on $[\alpha, \beta]$

$$|P_n(e^{i\theta})| \le B_{11}, \qquad n = 0, 1, 2, \dots \tag{21}$$

If on $[\alpha, \beta]$ one of the conditions

$$p(\theta) \le B_1, \qquad \omega_2^2(t,p)\frac{1}{t}\ln\frac{1}{t} \in \mathcal{L}_1; \tag{22}$$

$$p(\theta) \in \mathcal{L}_q, \qquad \frac{1}{p(\theta)} \in \mathcal{L}_{q'}, \qquad \omega_q(t,p)\frac{1}{t}\ln\frac{1}{t} \in \mathcal{L}_1; \tag{23}$$

$$p(\theta) \le B_1, \qquad \frac{\omega_q^q(t,p)}{t} \in \mathcal{L}_1, \qquad 1 < q \le 2; \tag{24}$$

$$\frac{\omega_1(t,p)}{t} \in \mathcal{L}_1, \tag{25}$$

is satisfied, then for almost all $\theta \in [\alpha', \beta']$, (3) holds.

The proof of Theorem 6 is based on the following lemma:

Lemma. Suppose that a weight $0 \leq \varphi(\theta) \leq B_{12}$ satisfies condition (1), and the corresponding system of orthonormal polynomials $\{\Phi_n(e^{i\theta})\}_0^\infty$ is uniformly bounded on $[0, 2\pi]$. Then at a point $e^{i\theta_0}$ of the unit circle we have

$$\lim_{n\to\infty} \Phi_n^*(e^{i\theta_0}) = \pi(e^{i\theta_0}, \varphi).$$

If the new weight $\psi(\theta) = \varphi(\theta)$ for $\theta \in [\alpha, \beta]$ and $\psi(\theta) \in \mathcal{L}_1$, $\psi^{-1}(\theta) \in \mathcal{L}_2$, then for the corresponding system of orthonormal polynomials $\{\Psi_n(e^{i\theta})\}_0^\infty$ we have

$$\lim_{n\to\infty} \Psi_n^*(e^{i\theta}) = \pi(e^{i\theta}, \psi), \qquad \theta \in [\alpha', \beta'].$$

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Note: Figure translations are in progress. See original paper for figures.

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