

ON THE PROBLEM OF THE OBLIQUE DERIVATIVE

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Abstract

Full Text

MATHEMATICS

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ON THE PROBLEM OF THE OBLIQUE DERIVATIVE

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The note is devoted to the following boundary-value problem:

Find a function w , regular and harmonic in the ball $\Sigma : \{x^2 + y^2 + z^2 < 1\}$, satisfying on the sphere $S : \{x^2 + y^2 + z^2 = 1\}$ the condition

$$\alpha(t, z)(x \partial w / \partial x + y \partial w / \partial y) + \beta(t, z) \partial w / \partial z = \chi, \quad (1)$$

where $t = x^2 + y^2$; α, β, χ are continuous Hölder functions defined on S , and α and β have no common zeros on S , and $\beta(0, 1) \cdot \beta(0, -1) > 0$.

We shall assume the desired function $w(x, y, z)$ to be Hölder continuous together with its first derivatives in the closed ball $\bar{\Sigma}$.

The function w can be represented inside the ball Σ by the absolutely and uniformly convergent series ⁽¹⁾:

$$w = \sum_{l=0}^{\infty} w_l^{(1)} t^{l/2} \cos l\varphi + \sum_{l=0}^{\infty} w_l^{(2)} t^{l/2} \sin l\varphi, \quad (2)$$

where $\varphi = \arctg y/x$, and the functions $w_l^{(1)}$ and $w_l^{(2)}$ are solutions of the differential equation

$$4t (\partial^2 u_l / \partial t^2) + \partial^2 u_l / \partial z^2 + 4(t+1) \partial u_l / \partial t = 0, \quad (3)$$

regular in the domain $D : \{0 < t < 1 - z^2, -1 < z < 1, t > 0\}$.

Lemma. *Every function χ , Hölder continuous on the sphere S , can be represented by the uniformly convergent series*

$$\chi = \sum_{l=0}^{\infty} A_l^{(1)}(\theta) t^{l/2} \cos l\varphi + \sum_{l=0}^{\infty} A_l^{(2)}(\theta) t^{l/2} \sin l\varphi. \quad (4)$$

Proof. As is known ⁽²⁾, every Hölder continuous function $\psi(\varphi)$ given on a circle can be represented by a Fourier series; therefore, for χ we have

$$\chi = \sum_{l=0}^{\infty} a_l(\theta) \cos l\varphi + \sum_{l=0}^{\infty} b_l(\theta) \sin l\varphi.$$

Construct a harmonic function $u(x, y, z)$, regular in the ball Σ , satisfying the condition $u = \chi$ on the sphere S . Every harmonic function regular in the ball Σ can be represented by an absolutely and uniformly convergent series in spherical functions ⁽¹⁾; rearranging and grouping terms in this series, we obtain

$$u = \sum_{l=0}^{\infty} B_l^{(1)}(r, \theta) t^{l/2} \cos l\varphi + \sum_{l=0}^{\infty} B_l^{(2)}(r, \theta) t^{l/2} \sin l\varphi.$$

Consider the value of the function u on the sphere $S_n : \{x^2 + y^2 + z^2 = 1 - 1/n\}$. For sufficiently large n we have $\max |u|_{S_n} - \chi| < \varepsilon$; applying Bessel's inequality to the function $u|_{S_n} - \chi$, we obtain

$$\sum_{l=0}^{\infty} \left[a_l(\theta) - B_l^{(1)}\left(1 - \frac{1}{n}, \theta\right) t^{l/2} \right]^2 + \sum_{l=0}^{\infty} \left[b_l(\theta) - B_l^{(2)}\left(1 - \frac{1}{n}, \theta\right) t^{l/2} \right]^2 < 2\pi\varepsilon^2,$$

hence

$$\lim_{n \rightarrow \infty} B_l^{(1)}\left(1 - \frac{1}{n}, \theta\right) t^{l/2} = a_l(\theta), \quad \lim_{n \rightarrow \infty} B_l^{(2)}\left(1 - \frac{1}{n}, \theta\right) t^{l/2} = b_l(\theta).$$

Substituting (2) and (4) into condition (1) and comparing the coefficients of $t^{l/2} \cos l\varphi$ and $t^{l/2} \sin l\varphi$, we obtain

$$2\alpha(t, z)t \partial u_l^{(i)} / \partial t + \beta(t, z) \partial u_l^{(i)} / \partial z + l\alpha(t, z)u_l^{(i)} = A_l^{(i)}(z), \quad (5)$$

$$i = 1, 2; \quad z = \cos \theta.$$

Thus problem (1) has been reduced to the following problem:

Find a solution, regular in the domain $D : \{0 < t < 1 - z^2, -1 < z < 1, t > 0\}$, of equation (3), satisfying condition (5) on the part of the boundary $L : \{t = 1 - z^2, -1 < z < 1\}$ of the domain D .

Consider problem (5). In [3] the following problem was studied:

Find a solution, regular in the domain D , of equation (3), satisfying on the curve L the condition

$$2\alpha(t, z)t \partial u_l / \partial t + \beta(t, z) \partial u_l / \partial z = A_l(z). \quad (6)$$

Problem (6) is solved by reduction to an integral equation. Problem (5) is also solved by reduction to an integral equation, and the equation for problem (5) differs from the integral equation for problem (6) by a completely continuous operator. Therefore the difference between the number of linearly independent solutions of the homogeneous problem and the number of necessary and sufficient orthogonality conditions on the function $A_l(z)$ for solvability of the nonhomogeneous problem is the same for problems (5) and (6).

Let $2\pi n$ be the increment of the argument of the function $\alpha\sqrt{t}-i\beta$ in traversing the curve L from the point $(0, 1)$ to the point $(0, -1)$. For problem (6), it was established in [3] that if $n \geq 0$, then the homogeneous problem has $2n+1$ linearly independent solutions, and the nonhomogeneous problem is always solvable; if $n < 0$, then the homogeneous problem has one linearly independent solution, and for solvability of the nonhomogeneous problem it is necessary and sufficient that $2n$ orthogonality conditions on the function $A_l(z)$ be fulfilled.

Theorem 1. The difference between the number of linearly independent solutions of the homogeneous problem (5) and the number of necessary and sufficient orthogonality conditions on the function $A_l(z)$ for solvability of the nonhomogeneous problem (5) is equal to $2n+1$ for all l .

Let $u_l^{(1)}, \dots, u_l^{(k_l)}$ be a complete orthonormal system of solutions of the homogeneous problem (5); let $q_l^{(1)}, \dots, q_l^{(j_l)}$ be the orthogonality functions to which the functions $A_l(z)$ must be orthogonal, necessarily and sufficiently, for solvability of the nonhomogeneous problem (5). Denote by H_S the space of functions Hölder-continuous on S , and by G_Σ the space of functions harmonic and regular in the ball Σ , whose first derivatives satisfy a Hölder condition in the closed ball $\bar{\Sigma}$. Consider the subspace K_Σ of the space G_Σ generated by the functions

$$u_l^{(1)}t^{l/2} \cos l\varphi, u_l^{(1)}t^{l/2} \sin l\varphi, \dots, u_l^{(k_l)}t^{l/2} \cos l\varphi, u_l^{(k_l)}t^{l/2} \sin l\varphi, l = 0, 1, \dots$$

We also consider the subspace R_S of the space H_S , generated by the functions $q_1^{(1)}t^{l/2} \cos l\varphi, q_1^{(1)}t^{l/2} \sin l\varphi, \dots, q_1^{(j_l)}t^{l/2} \cos l\varphi, q_1^{(j_l)}t^{l/2} \sin l\varphi, l = 0, 1, \dots$. K_Σ is the space of solutions of the homogeneous problem (1), and for the solvability of the nonhomogeneous problem (1) it is necessary that the function χ be orthogonal to the space R_S .

Theorem 2. *If $n \geq 0$, then the homogeneous problem (1) has infinitely many linearly independent solutions.*

This theorem is a consequence of Theorem 1.

Let $\alpha(t, z)$ and $\beta(t, z)$ be polynomials. Extend the coefficients $\alpha(t, z)$ and $\beta(t, z)$ into the ball Σ in the following way: since on the sphere S we have $t = 1 - z^2$, set

$$P(z) = \alpha(1 - z^2, z), \quad Q(z) = \beta(1 - z^2, z)$$

in Σ . The left-hand side of the boundary condition (1), after continuation, has the form

$$P(z)(x \partial w / \partial x + y \partial w / \partial y) + Q(z) \partial w / \partial z. \quad (1')$$

In this case problem (1) is a special case of the problem studied in (4).

We study the singular points of the vector field

$$\Omega : \{xP(z), yP(z), Q(z)\}.$$

The singular points of the field Ω are the points $(0, 0, z_0)$, $Q(z_0) = 0$. The type of a singular point is determined by the roots of the equation

$$\begin{vmatrix} P(z) - \lambda & 0 & 0 \\ 0 & P(z) - \lambda & 0 \\ 0 & 0 & Q'(z) - \lambda \end{vmatrix} \equiv [P(z) - \lambda]^2 [Q'(z) - \lambda] = 0.$$

If at the singular point $(0, 0, z_0)$ we have $P(z_0)Q'(z_0) > 0$, then this point is a node; if $P(z_0)Q'(z_0) < 0$, then the singular point $(0, 0, z_0)$ is a saddle. In (4) it is shown that in the case of a singular point of saddle type, problem (1) is unconditionally solvable. From the unconditional solvability of problem (1) follows the unconditional solvability of problem (5) for all natural l . Consequently, the space R_S consists only of the single element 0, and the homogeneous problem (5) has the same number $2n + 1$ of linearly independent solutions for all natural l .

Theorem 3. *If at all singular points $(0, 0, z_0)$ of the vector field Ω the relation $P(z_0)Q'(z_0) < 0$ holds, then the nonhomogeneous problem (1) is always solvable, and the homogeneous problem has infinitely many linearly independent solutions.*

Suppose further that $P(z)$ and $Q(z)$ are twice continuously differentiable functions, and that $P(z)$ and $Q(z)$ have no common zeros on the interval $-1 \leq z \leq 1$. Denote by σ the set of points of the sphere S at which the vector field Ω is tangent to the sphere S . In the work (5) it is shown that a solution of the homogeneous problem (1) can attain an extremum only at points of the set σ . In order that a point $x \in \sigma$ not be an extremum point of a nonconstant solution of the homogeneous problem (1) of the form $u_l(r, \theta) \cos l\varphi$, it is sufficient that at this point the inequality

$$z \frac{d}{dz} [P(z)(1 - z^2) + zQ(z)] / P(z) > 0 \quad (7)$$

hold.

If at all points of the set σ inequality (7) holds, then the homogeneous problem (1) has no nonzero solutions of the form $u_l(r, \theta) \cos l\varphi$, $l \geq 0$; $v_l(r, \theta) \sin l\varphi$, $l > 0$, and from the reduction of problem (1) to problem (5) it follows that the homogeneous problem (1) has no solutions distinct from the constant solution.

Theorem 4. *If at all points of the set σ inequality (7) holds, then the homogeneous problem (1) has no solutions distinct from a constant that possess continuous second derivatives in $\Sigma \cup S$.*

Remark. Similarly, one may consider the problem on the oblique derivative with the boundary condition

$$\alpha(t, z) (x \partial w / \partial x + y \partial w / \partial y + \beta(t, z) (\partial w / \partial z + \gamma(t, z) (y \partial w / \partial x - x \partial w / \partial y))) = \chi.$$

The role of problem (5) in this case is played by the following problem:

Find a solution of the system of equations

$$4t \partial^2 u_l^{(i)} / \partial t^2 + \partial^2 u_l^{(i)} / \partial z^2 + 4(l+1) \partial u_l^{(i)} / \partial t = 0, \quad i = 1, 2,$$

satisfying the conditions

$$2\alpha(t, z)t \partial u_l^{(1)} / \partial t + \beta(t, z) \partial u_l^{(1)} / \partial z + l\alpha(t, z)u_l^{(1)} - l\gamma(t, z)u_l^{(2)} = A_l^{(1)}(z),$$

$$2\alpha(t, z)t \partial u_l^{(2)} / \partial t + \beta(t, z) \partial u_l^{(2)} / \partial z + l\alpha(t, z)u_l^{(2)} + l\gamma(t, z)u_l^{(1)} = A_l^{(2)}(z)$$

on the curve L .

All the theorems carry over to this case, except Theorem 4.

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