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THEORY OF  
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**Abstract**

**Full Text**

**MATHEMATICAL PHYSICS**

**R. D. BACHELIS, V. G. MELAMED**

**ON THE NONUNIQUENESS OF STATIONARY SOLUTIONS FOR A SYSTEM OF EQUATIONS OF THE THEORY OF COMBUSTION WITH PIECEWISE CONSTANT RATE CONSTANT AND COEFFICIENTS OF THERMAL CONDUCTIVITY AND DIFFUSION**

*(Presented by Academician Ya. B. Zel'dovich on February 6, 1965)*

As shown in <sup>(1)</sup>, a one-dimensional process of combustion of a gas mixture is determined by a nonlinear system of the form

$$\lambda \frac{du}{dy} = \frac{d}{dy} \left[ \alpha(u) \frac{du}{dy} \right] + F(u)C; \quad (1)$$

$$\lambda \frac{dC}{dy} = \frac{d}{dy} \left[ \beta(u) \frac{dC}{dy} \right] - F(u)C \quad (2)$$

under the conditions

$$u(-\infty) = 0, \quad C(-\infty) = C_0 > 0, \quad C(\infty) = 0, \quad (3)$$

$u(\infty)$  exists and is finite. Here  $u \geq 0$  is the temperature of the mixture;  $C \geq 0$  is the concentration of the active substance;  $\lambda = \text{const} > 0$  is the velocity of propagation of the flame front;  $F(u) \geq 0$  is the reaction-rate constant;  $\alpha(u)$ ,  $\beta(u)$  are, respectively, the coefficients of thermal conductivity and diffusion.

Obviously, any solution of (1)–(3) admits an arbitrary parallel translation along the  $y$ -axis. The system (1)–(3) has the first integral

$$\lambda \{ C(y) + u(y) - C_0 - u(-\infty) \} = \alpha(u) \frac{du}{dy} + \beta(u) \frac{dC}{dy}, \quad (4)$$

which expresses the dependence between the energy flux and the state of the mixture at any point.

In the case of the existence of  $u'(\pm\infty)$ ,  $C'(\pm\infty)$ , it follows from this that

$$u(\infty) = C_0. \quad (5)$$

It is required to determine the functions  $C(y)$  and  $u(y)$  on  $(-\infty, \infty)$ , as well as the constant  $\lambda$ , for given  $F(u)$ ,  $\alpha(u)$ ,  $\beta(u)$ , and  $C_0$ .

In (2) it was proved that, in the case of the so-called “cut-off” rate constant\* with  $\alpha = \text{const}$  and  $\beta = \text{const}$ , the solution of the problem under consideration is unique when  $\alpha/\beta \geq 1$ . In the opposite case the question remained open. In the present work it is shown that, indeed, when  $\alpha/\beta < 1$  nonuniqueness of the solution of the system (1)–(3) is possible for piecewise constant  $F(u)$ ,  $\alpha(u)$ , and  $\beta(u)$ , with  $F(u)$  cut off. The aim of the paper is not to give an algorithm for finding  $\lambda$  and to investigate the question of the number of solutions, but rather to construct a corresponding example of a problem for which three solutions are found.

Let  $n$  be the number of regions in each of which  $F(u)$ ,  $\alpha(u)$ ,  $\beta(u)$  are constants. Obviously, for  $n = 1$  (1)–(3) has no solution. Hence  $n \geq 2$  is necessary. Specify values  $u_k$ ,  $k = 0, 1, 2, \dots, n$ , so that

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\* That is,  $F(u) = 0$  on an interval one of whose ends is  $u(-\infty)$ . Such a specification of  $F(u)$  excludes a continuum of solutions of (1)–(3) (3).

in addition,  $u_0 = 0$ ,  $u_n = C_0$ ,  $u_{k-1} < u_k$ . On each interval  $(u_{k-1}, u_k)$  set

$$\begin{aligned} \alpha(u) &\equiv \text{const} = \alpha_k > 0, & \beta(u) &\equiv \text{const} = \beta_k > 0, \\ F(u) &\equiv \text{const} = F_k, \end{aligned} \quad (6)$$

$F_k > 0$  for  $k > 1$ ,  $F_1 = 0$ . Everywhere below the functions  $F(u)$  and  $\alpha(u)$ ,  $\beta(u)$  in the system (1), (2) will be regarded as prescribed in the indicated way.

We shall call the values  $u_k$ ,  $k = 1, 2, \dots, n - 1$ , points of gluing for the system (1), (2). We seek a constant  $\lambda > 0$  and functions  $u(y)$  and  $C(y)$  on  $(-\infty, \infty)$  satisfying the system (1), (2) everywhere outside the gluing points, and also the conditions (3). It is easy to prove that  $u(y)$  increases strictly for all  $y$ .

Consider the gluing points. Let  $u(y_k) = u_k$ . From the increase of  $u(y)$  follows the uniqueness of  $y_k$  for a fixed solution and a given  $k$ . From the continuity of  $u(y)$  and  $C(y)$  at  $y = y_k$  we have

$$u(y_k - 0) = u(y_k + 0) = u_k, \quad C(y_k - 0) = C(y_k + 0) = C_k. \quad (7)$$

In addition, in view of the absence at the gluing points of sources of heat and of substance, we have

$$\alpha_k u'(y_k - 0) = \alpha_{k+1} u'(y_k + 0), \quad \beta_k C'(y_k - 0) = \beta_{k+1} C'(y_k + 0). \quad (8)$$

From (7) and (8) there also follows the fulfillment of (4), (5) for the  $F(u)$  and  $\alpha(u)$ ,  $\beta(u)$  under consideration.

Obviously, any solution of (1), (2) under the conditions (3), (5)–(8) admits an arbitrary parallel translation along the  $y$ -axis. For

$$u \in (u_{k-1}, u_k), \quad k > 1, \quad \lambda^2(\beta_k - \alpha_k) \neq F_k \alpha_k^*$$

the general solution of (1), (2) has the form:

For  $k > 1$

$$\begin{aligned} u(y) &= A_{1,k} \exp m_{1,k} y + A_{2,k} \exp m_{2,k} y + A_{3,k} \exp \frac{\lambda y}{\alpha_k} + A_{4,k}, \\ C(y) &= \frac{1}{F_k} [m_{1,k}(\lambda - \alpha_k m_{1,k}) A_{1,k} \exp m_{1,k} y + \\ &\quad + m_{2,k}(\lambda - \alpha_k m_{2,k}) A_{2,k} \exp m_{2,k} y], \end{aligned} \quad (9)$$

where

$$m_{1,k} = \frac{\lambda}{2\beta_k} (\lambda - \sqrt{\lambda^2 + 4\beta_k F_k}), \quad m_{2,k} = \frac{1}{2\beta_k} (\lambda + \sqrt{\lambda^2 + 4\beta_k F_k}).$$

For  $k = 1$ , by virtue of  $F_1 = 0$ , the general solution of (1), (2) has the form

$$u = A_{3,1} \exp \frac{\lambda}{\alpha_k} y + A_{4,1}, \quad C = A_{1,1} \exp \frac{\lambda}{\beta_k} y + A_{2,1}. \quad (10)$$

Here  $A_{ik}$ ,  $i = 1, 2, 3, 4$ , are constants of integration.

Let  $\lambda > 0$  be given. From (3) it follows that  $A_{2,n} = A_{3,n} = 0$ . From (5) we have  $A_{4,n} = C_0$ . From the requirement of monotone increase of  $u(y)$  it follows that  $A_{1,n} < 0$ . The latter entails a monotone decrease of  $C(y)$ . The arbitrariness in the choice of  $A_{1,n}$  is explained by the possibility of an arbitrary parallel translation along the  $y$ -axis.

Thus,  $u(y)$  and  $C(y)$  are determined for all those  $y$  for which  $u(y) \in (u_{n-1}, u_n)$ . We find  $y_{n-1}$  from the condition  $u(y_{n-1}) = u_{n-1}$ . From (7), (8) we determine  $A_{i,n-1}$ ,  $i = 1, 2, 3, 4$ , then  $y_{n-2}$ , and so on. It can be proved that

\* The last inequality, while considerably simplifying the writing of the solution, does not substantially restrict the class of piecewise constant functions  $F(u)$  and  $\alpha(u)$ ,  $\beta(u)$ .

**Table 1**

$$F_1 = 0; \quad F_2 = F_3 = 0, 1; \quad F_4 = 1249, 687500; \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, 0; \quad \beta_1 = \beta_2 = 100, 0 \quad \beta_3 = 16, 6666$$

$\lambda$	$y \in (-\infty; y_1) y_1$	$y \in (y_1; y_2) y_2$	$y \in (y_2; y_3) y_3$	$y \in (y_3; \infty)$
$\lambda = y$ 1, 000000	$(-\infty; -6, 279258)$	$(-6, 279258; 8, 831657)$	$(8, 831657; 10, 583165)$	$(10, 583165; \infty)$
$\lambda = u$ 1, 000000	$0, 307124 \exp(-24, 748215y)$	$10, 842086$	$24, 748125$	$0, 307124 \exp(-24, 748215y)$
$\lambda = C$ 1, 000000	$27, 126978$	$24, 748125$	$10, 842086$	$0, 307124 \exp(-24, 748215y)$
$\lambda = \alpha u'$ 1, 000000		$0, 307124$	$10, 842086$	$24, 748125$
$\lambda = \beta C'$ 1, 000000		$-24, 077115$	$-24, 101825$	$-25, 238223$
$\lambda = y$ 1, 0345919	$(-\infty; -8, 945525)$	$(-8, 945525; 8, 485981)$	$(8, 485981; 10, 583165)$	$(10, 583165; \infty)$
$\lambda = u$ 1, 0345919	$0, 307124 \exp(-24, 739565y)$	$10, 352225$	$24, 739565$	$0, 307124 \exp(-24, 739565y)$
$\lambda = C$ 1, 0345919	$27, 126978$	$24, 739565$	$10, 352225$	$0, 307124 \exp(-24, 739565y)$
$\lambda = \alpha u'$ 1, 0345919		$0, 317748$	$10, 352225$	$24, 739565$
$\lambda = \beta C'$ 1, 0345919		$-24, 227978$	$-24, 045675$	$-25, 246266$
$\lambda = y$ 2, 000000	$(-\infty; -30, 022778)$	$(-30, 022778; 0, 22778)$	$(0, 22778; 10, 583165)$	$(10, 583165; \infty)$
$\lambda = u$ 2, 000000	$0, 307124 \exp(-24, 462663y)$	$0, 22778$	$10, 583165$	$0, 307124 \exp(-24, 462663y)$
$\lambda = C$ 2, 000000	$27, 126978$	$0, 22778$	$10, 583165$	$0, 307124 \exp(-24, 462663y)$
$\lambda = \alpha u'$ 2, 000000		$0, 614248$	$0, 416001$	$24, 501875$
$\lambda = \beta C'$ 2, 000000		$-27, 724513$	$-12, 974549$	$-25, 462663$

For all the remaining intervals, likewise,  $u(y)$  will increase monotonically, while  $C(y)$  will decrease monotonically. At the same time, from (4), (5), and (9) it is

easy to obtain that  $A_{4,k} = C_0$  for all  $k > 1$ . Completing the described process, we find the constants entering into (10). If, in doing so,  $A_{4,1} = 0$  is obtained, then, by virtue of (4),  $A_{2,1} = C_0$ , and consequently the prescribed  $\lambda$  and the found  $u(y)$  and  $C(y)$  are solutions of the posed problem.

The authors have established that for  $n = 2$  the solution (up to a parallel translation along the  $y$ -axis) is unique. In the case  $n = 3$ , the construction of an example of nonuniqueness of the solution encountered difficulties because of the small number of degrees of freedom. For  $n = 4$ , the values of  $F_k$ ,  $C_0$ ,  $\alpha_k$ ,  $\beta_k$ , and also the three corresponding solutions of the problem under consideration are given (see Table 1).

In constructing the example, the inverse problem was solved, i.e., two different values of  $\lambda$ ,  $\lambda = 1$  and  $\lambda = 2$ , were prescribed in advance, while  $F_k$ ,  $\alpha_k$ ,  $\beta_k$ ,  $u_k$  were chosen so that the prescribed  $\lambda$  and the found  $u(y)$  and  $C(y)$  satisfied (1)–(3), (5)–(8). In the process of solving the inverse problem, a system of two first-order equations was used, obtained, with account of (4), from (1)–(3) as a result of the substitution  $a(u) du/dy = v$ . In addition, investigation of the latter made it possible to discover that there always exists an odd number of solutions of (1)–(3). In particular, after solving the inverse problem considered, one more solution was found, corresponding to  $\lambda = 1.0345919$ . In all probability, the values of  $\lambda$  corresponding to stable and unstable solutions alternate.

In the case where the conditions under which uniqueness is proved are not fulfilled, the question of the number of solutions is a subject for special investigation.

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