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V. L. KLYUSHIN

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Abstract

Full Text

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ON A CERTAIN CLASS OF PREIMAGES OF METRIC SPACES

(Presented by Academician P. S. Aleksandrov, January 16, 1965)

A continuous mapping $f : X \to R$ is called, as is known, an ω-mapping for a cover ω of a space X, if for every point $y \in R$ there exists a neighborhood Oy such that the set $f^{-1}Oy$ is contained in one of the elements of the cover ω. We shall call a continuous mapping $f : X \to R$ a weak ω-mapping for the cover ω, if the preimage $f^{-1}y$ of each point $y \in R$ is contained in at least one of the elements of the cover ω.

In the present paper we consider spaces that admit, for every open cover ω, weak ω-mappings into metric spaces; in particular, spaces compactifiable in metric spaces and spaces that are perfect preimages of metric spaces. As B. A. Pasynkov showed ([1]), the spaces under consideration are spectrally metrizable, i.e. can be obtained as limits of inverse spectra of metric spaces.

A sequence of covers $\{\omega_i\}$ of a space X is called normal if, for every $i = 1, 2, ...$, the cover ω_{i+1} is star-refined into the cover ω_i, i.e. the star $O_{\omega_{i+1}}x$ of each point $x \in X$ with respect to the cover ω_{i+1} is contained in some element of the cover ω_i. We shall say that a cover β is regularly refined into a cover α if, for any two intersecting elements of the cover β, there exists an element of the cover α containing their union; a sequence of covers $\{\omega_i\}$ will be called regular if, for every $i = 1, 2, ...$, the cover ω_{i+1} is regularly refined into the cover ω_i.

Theorem 1. *Let ω be some cover of a space X. In order that there exist a weak ω-mapping of the space X into some metric space, it is necessary and sufficient that either of the following conditions be satisfied:*

1°. *There exists a regular sequence of covers $\{\omega_i\}$ of the space X such that the system of sets*

$$\left\{\bigcap_{i=1}^{\infty} O_{\omega_i}x\right\}_{x\in X}$$

is refined into the cover ω.

2°. *There exists a normal sequence of covers $\{\omega_i\}$ of the space X such that the system of sets*

$$\left\{\bigcap_{i=1}^{\infty} O_{\omega_i}x\right\}_{x\in X}$$

is refined into the cover ω.

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For the proof of Theorem 1 the same construction is used as in the work of V. I. Ponomarev ([2]).

Theorem 2. *Let the space X be countably paracompact and collectionwise normal, and let ω be some cover of it. In order that there exist a weak ω-mapping of the space X into some metric space, it is necessary and sufficient that there exist a sequence of locally finite covers $\{\omega_i\}$ such that, for every $i = 1, 2, \ldots$, the cover ω_{i+1} is refined into ω_i and the system of sets*

$$\left\{\bigcap_{i=1}^{\infty} O_{\omega_i} x\right\}_{x \in X}$$

is refined into ω.

In view of the aforementioned result of B. A. Pasynkov, for the approximation

spaces as inverse spectra of metric spaces, it is obviously sufficient that for every cover of it any one of the conditions formulated above be satisfied.

Theorem 3. *A countably compact space X is bicompact if, for every cover ω of it, there exists a weak ω-mapping $f : X \to R$ into some metric space R.*

From Theorem 1 and the theorem of A. S. Parkhomenko on compactifications it follows that

Theorem 4. *In order that there exist a compactification of the space X into a metric space, it is necessary and sufficient that there exist a countable sequence of covers satisfying the following conditions:*

1) $$\bigcap_{i=1}^{\infty} O_{\omega_i} x = x$$

   for every point $x \in X$.

2) *For each cover ω_i there exists a weak ω_i-mapping $f : X \to R_i$ into a metric space.*

Let a cover ω be given. Put $\omega^* = \{O_\omega x\}_{x \in X}$. The following is readily proved.

Lemma. *If in every cover of the space X one can properly inscribe some cover, then for every cover ω there exists an ω^*-mapping $f : X \to R$ into a metric space.*

With the aid of this assertion one proves

Theorem 5. *If in every cover ω of the space X one can properly inscribe a point-countable cover, then the space X is paracompact (and, consequently, admits an ω-mapping into a metric space).*

Consider the case when the existence of a weak ω-mapping implies the existence of an ω-mapping.

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Theorem 6. *Let the space X be complete in the sense of Čech, and let in every cover of it one be able to properly inscribe some cover. If for a cover ω of the space X there exists a weak ω-mapping $f : X \to R$ into a metric space, then there exists a perfect ω-mapping $f_1 : X \to R_1$ into a complete metric space.*

The proof of Theorem 6 is based on the criterion of completeness in the sense of Čech, due to A. V. Arhangel' skii (3).

Finally, consider spaces for which there exist perfect mappings into metric spaces. Following A. V. Arhangel' skii, we shall call them paracompact p-spaces. It has been proved by the author (4) that for every cover ω of a paracompact p-space X there exists a perfect ω-mapping into a metric space. A more general assertion holds.

Theorem 7. *For every countable sequence of covers $\{\omega_i\}$ of a paracompact p-space X (of dimension $\dim X = n$) there exists a metric space R (of dimension $\dim R = n$) and a mapping $f : X \to R$ which is a perfect ω_i-mapping for each cover ω_i in this sequence.*

Suppose the sequence of covers $\{\omega_i\}$ satisfies the following condition: if one element is taken from each cover of this sequence, then the intersection of these sets contains at most one point. Then the mapping f, which is an ω_i-mapping for every ω_i, $i = 1, 2, ...$, is a compactification. Therefore, from Theorem 7 there follows the following theorem, due to A. V. Arhangel' skii:

Theorem 8. *A paracompact p-space that is compactifiable into a metric space is metrizable.*

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CITED LITERATURE

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2. V. Ponomarev, DAN, 141, No. 3, 561 (1961).
3. A. V. Arhangel' skii, Vestn. Mosk. Univ., Ser. Math., Mech., No. 2, 37 (1961).
4. V. Klyushin, DAN, 159, No. 4 (1964).

Note: Figure translations are in progress. See original paper for figures.

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