I. A. BEREZHNOI, D. D. IVLEV

1965

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

THEORY OF ELASTICITY

I. A. BEREZHNOI, D. D. IVLEV

ON THE INFLUENCE OF VISCOSITY ON THE MECHANICAL BEHAVIOR OF PLASTIC MEDIA

(Presented by Academician A. Yu. Ishlinskii, November 16, 1964)

Recently a number of works have appeared devoted to the theory of visco-elasto-plastic media, among which we note (1−5).

In the present work, a generalization of the model of an anisotropically hardening elastic-plastic body (6−8) is considered by introducing an internal viscosity element that determines the relaxation of residual microstresses. It is shown that, for an anisotropically hardening plastic material, the manifestation of the relaxing properties of microstresses makes it possible to explain the appearance of "corner points" when the loading trajectory is broken.

Possible combinations of the elements of elasticity, viscosity, and plasticity are shown in Fig. 1. The presence of viscosity makes the mechanical behavior dependent on time; therefore the behavior of the material will depend on the rate of loading. Two limiting cases should be distinguished: the loading rate is infinitely small; the loading rate is infinitely large (instantaneous loading). The inertial properties of the models are not taken into account here.

Fig. 1

Under infinitely slow loading, relaxation processes occur completely, and the viscosity element offers no resistance to the forces. Conversely, under an instantaneous change of forces, the viscosity element behaves like a rigid link. Similarly, the material behaves with an arbitrarily small or arbitrarily large coefficient of viscosity.

For given constants characterizing the mechanical properties of the models, the stress-strain dependence $(\sigma - e)$ depends on the loading rate. Figure 2 gives the regions of dependence $(\sigma - e)$ corresponding to the models shown in Fig. 1. To the model shown in Fig. 1a there corresponds, in Fig. 2, the region bounded by the lines *oa*, *obd*; to the model of Fig. 1b—the region bounded by the lines *obd*,

Fig. 2 and Fig. 3

Figure 2: Fig. 2 and Fig. 3

ocbd; to the model of Fig. 1c—the region bounded by the lines *ocd*, *ocf*; to the model of Fig. 1d—the region bounded by the lines *ocf*, *ocg*.

Let us focus attention on the model of Fig. 1c. In the case when the loading is instantaneous or the coefficient of viscosity is unboundedly large, the model of an anisotropically hardening plastic material applies; when the loading is infinitely slow or the coefficient of viscosity is equal to zero—the model of an ideally plastic body applies. We also note that with unbounded growth of the stiffness coefficient of the elastic spring, the link between the viscosity element and plasticity becomes rigid, and the model of a viscoplastic body (a Bingham body), etc., applies.

Following the ideas of (^{8}), in Fig. 3 we present a useful two-dimensional model corresponding to the one-dimensional model of Fig. 1c. Let T_1 and T_2 denote the external forces acting on the plastic element of Fig. 3; let s_1, s_2 denote the forces in the springs. Further, let q_1, q_2 denote the displace-

of the plastic element; by r_1, r_2, the displacements of the viscous element. The complete system of equations determining the mechanical behavior of the model under consideration will have the form

$$(T_1 - s_1)^2 + (T_2 - s_2)^2 = k^2, \qquad \Delta q_1 : \Delta q_2 = (T_1 - s_1) : (T_2 - s_2),$$

$$s_i = c(q_i - r_i), \qquad s_i = \mu \dot{r}_i, \qquad i = 1, 2, \tag{1}$$

where k is the coefficient of friction; μ is the coefficient of viscosity; c is the stiffness coefficient; Δq_i are increments of displacement of the plastic element; a superposed dot denotes differentiation with respect to time.

Fig. 2 Fig. 3

When dynamic analogies are used, stresses are put in correspondence with forces, and strains with displacements. Denote by σ_{ij} the tensor of actual stresses (corresponding to the forces T_i), by s_{ij} (following the terminology of [8]) the tensor of microstresses (corresponding to the forces s_i), and by e_{ij} the tensor of actual strains (corresponding to the displacements q_i). We shall assign primes to the deviators of the corresponding tensors. We note that the medium under consideration is rigid-plastic; therefore e_{ij} are the components of plastic strain.

We write the yield condition in the form

$$f(\sigma_{ij} - s_{ij}) = 0. \tag{2}$$

Assume that the function f is sufficiently smooth. Consider the expression for the increment of work $dA = \sigma_{ij}\, de_{ij}$. Taking as valid Mises' principle [9] on the extremality of the work of stresses, for given strain increments of the medium, among all possible values of the stress components, we obtain the associated flow law

$$de_{ij} = d\lambda\, \partial f/\partial \sigma_{ij}, \qquad d\lambda \geq 0. \tag{3}$$

Starting from Mises' principle, one can also establish the convexity of the loading function $f = 0$.

For simplicity, assume that the loading function does not depend on the first invariants of σ_{ij}, s_{ij}. Next, set

$$s'_{ij} = c(e'_{ij} - \chi'_{ij}), \qquad s'_{ij} = \mu \dot{\chi}'_{ij}, \qquad e_{ij} = \chi_{ij} = 0, \qquad c, \mu = \text{const}. \tag{4}$$

The system of relations (2)–(4) completely determines the mechanical properties of the body model under consideration, whose index is Pev, according to the classification [5].

Let us note the obvious properties of the model under consideration. The material is rigid-plastic. If the material is brought beyond the yield limit and the loads are fixed, then a creep process will occur. If the material is brought beyond the yield limit and then unloaded, then, owing to relaxation of the microstresses s_{ij}, the loading surface will tend in time to occupy its initial position, etc.

The known definitions of loading fully retain their validity: loading occurs when $f = 0,\ \dot{f} > 0$; unloading, when $f < 0,\ \dot{f} < 0$; neutral loading, when $f = 0,\ \dot{f} = 0$.

Let us consider the case of neutral loading. It follows from (2) that in this case

$$\frac{\partial f}{\partial \sigma_{ij}} \left(\dot{\sigma}_{ij} - \dot{s}'_{ij} \right) = 0. \tag{5}$$

Under neutral loading the increments of plastic strains are equal to zero: $de'_{ij} = 0,\ d\lambda = 0$. Differentiating the first relation (4) with respect to time, for the cases of neutral loading and unloading we obtain

$$\dot{s}'_{ij} = -c\chi'_{ij}. \tag{6}$$

From the second relation (4) and (6) we find

$$\dot{s}'_{ij} + \frac{c}{\mu} s'_{ij} = 0, \tag{7}$$

Fig. 4

Figure 3: Fig. 4

whence

$$s'_{ij} = s'^{0}_{ij} \exp\left(-\frac{c}{\mu}t\right),$$

$$\dot{s}'_{ij} = -\frac{c}{\mu} s'^{0}_{ij} \exp\left(-\frac{c}{\mu}t\right), \tag{8}$$

where s'^{0}_{ij} are the microstresses at the time $t = 0$, measured from the start of unloading or neutral loading. Thus, during unloading and neutral loading the relaxation rates of the internal microstresses are always proportional to the microstresses s'^{0}_{ij}.

Fig. 4

According to (5), in stress space under neutral loading the vector $\vec{\sigma} - \dot{\mathbf{s}}$, with components $\dot{\sigma}_{ij} - \dot{s}'_{ij}$, is orthogonal to the normal $\mathbf{n}$ to the loading surface $f = 0$. Figure 4 shows a regular loading surface $f = 0$. At the given point σ_{ij} of the loading surface a normal $\mathbf{n}$ is drawn and the vector $-\dot{\mathbf{s}}$ is constructed. The vector of the rate of change of the actual stresses $\dot{\vec{\sigma}}$ under neutral loading is directed inward with respect to the loading surface, so that the vector $\dot{\vec{\sigma}} - \dot{\mathbf{s}}$ lies in the tangent plane to the surface $f = 0$.

Consequently, at the given point of a regular loading surface, owing to relaxation of the microstresses, the experimenter may observe the effect of an "angular point" on the loading surface: an increment of plastic strains may occur also in the case when the stress-increment vector is directed inward with respect to the initial loading surface. Similar effects are noted in work (4).

The angle formed by the vector $\dot{\vec{\sigma}}$ with the normal $\mathbf{n}$ obviously depends on the rate of loading $\dot{\sigma}$. In the case of an instantaneous change of stresses $\dot{\vec{\sigma}} \gg \dot{\mathbf{s}}$, and the vector $\dot{\vec{\sigma}}$ is orthogonal to $\mathbf{n}$.

In the present case, not only the effect of an internal but also that of an external angular point may be observed. Indeed, if by an instantaneous change of the stress vector $\vec{\sigma}$ one passes to some other point of the loading surface (σ_1 in Fig. 4), then the vectors $\mathbf{s}$ and $-\dot{\mathbf{s}}$ retain their initial value. On a closed loading surface there are always such points for which the vector $-\dot{\mathbf{s}}$ is directed inward with respect to the loading surface. Then the vector $\dot{\vec{\sigma}}$, corresponding to neutral loading,

will lie, generally speaking, outside the instantaneous loading surface and will form an acute angle with the normal $\mathbf{n}$ at the given point of the loading surface.

Thus, the relaxation effects of the microstresses s_{ij} ensure the tendency of the loading surface $f(\sigma_{ij} - s_{ij}) = 0$ toward the initial state in loading space, which makes plastic deformation possible when the increment of the stress vector $\Delta\vec{\sigma}$ lies either inside or outside the region bounded by the loading surface of the preceding state. This latter circumstance explains the apparent appearance of an angular point on the loading surface, on the basis of the concept of relaxing microstresses.

Let us note that, for the model under consideration, the relation between stresses and strains in the example of pure shear can be written in the form

$$c\tau + \mu\dot{\tau} + c\mu\dot{e} = \text{const}. \tag{9}$$

Consequently, the loading function in this case also depends on the rate of change of the stresses.

The authors express their gratitude to V. D. Klyushnikov for a number of valuable comments.

Voronezh State
University

Received
4 XI 1964

REFERENCES

1. H. Landau, J. Weiner, E. Zwicky, J. Appl. Mech., **27**, 481 (1960).
2. D. D. Ivlev, In collection: *Problems of the Mechanics of Continuous Media*, in honor of Acad. N. I. Muskhelishvili, Publishing House of the USSR Academy of Sciences, 1961.
3. P. P. Rezyna, Techn. Rep., Brown Univ., No. 76 (1962).
4. P. M. Naghdi, S. A. Murch, Applied Mechanics, **30**, No. 3 (1963).
5. D. D. Ivlev, DAN, **148**, No. 1 (1963).
6. W. Prager, ZAMM, **15**, H. 1/2 (1935).
7. A. Yu. Ishlinskii, Ukr. Math. Journal, **6**, No. 3 (1954).
8. Yu. I. Kadashevich, V. V. Novozhilov, Applied Mathematics and Mechanics, **22**, issue 1 (1958).
9. R. Mises, ZAMM, **8**, H. 1 (1928).