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# MATHEMATICS

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1965

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**Abstract**

**Full Text**

## MATHEMATICS

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### ON NONLINEAR PARABOLIC EQUATIONS HAVING NONDIVERGENT FORM

*(Presented by Academician I. G. Petrovskii on January 4, 1965)*

Let  $Q = \{a \leq x \leq b, 0 \leq t \leq T\}$  be a rectangle with lateral boundary  $\Gamma$ ;  $u(x, t)$  a function defined in  $Q$ ;  $u^{(s)} \equiv d^s u / dt^s$ ,  $D^l u \equiv \partial^l u / \partial x^l$ . We shall consider known the following spaces:

$$W_p^{(l)}(a, b) \equiv W_p^{(l)}, \quad \dot{W}_p^{(l)}(a, b) \equiv \dot{W}_p^{(l)}, \quad \dot{W}_{p'}^{(-l)}(a, b) \equiv \dot{W}_{p'}^{(-l)} \quad (p > 1, p' = p/(p-1)).$$

Further,  $W_{p,q}^{(l,1)}$  denotes the subspace of  $W_{p,q}^{(l,1)}$  such that, for  $u(x, t) \in W_{p,q}^{(l,1)}$ ,

$$u(x, 0) = 0, \quad u|_{\Gamma} = 0, \dots, \quad D^{l-1}u|_{\Gamma} = 0;$$

$\dot{W}_{p,q}^{(l,1)}$  is the same, except that  $u(x, T) = 0$  ( $q \geq 1$ ). The norms of the spaces  $\mathcal{L}_p(0, T; \dot{W}_p^{(l)})$  and  $\mathcal{L}_p(0, T; \dot{W}_{p'}^{(-l)})$  will be denoted by  $\|u\|_{l,p}$  and  $\|u\|_{-l,p'}$ .

Let  $k \geq 0$ ,  $l \geq 0$ ,  $m \geq 1$  be integers, with  $k + l = 2m$ . Then:

a)  $H = H(l, k, r, p, p')$  is the closure of smooth functions  $u(x, t)$ ,

$$u(x, 0) = 0, \quad u|_{\Gamma} = 0, \dots, \quad D^{r-1}u|_{\Gamma} = 0, \quad D^{2r}u|_{\Gamma} = 0, \dots, \quad D^{l-1}u|_{\Gamma} = 0$$

in the norm

$$\|u\| = \|u\|_{l,p} + \|u'\|_{-k,p'},$$

where  $k \leq l$ ,  $2r = l - k$ ;

b)  $H_1 = H_1(l, k, r, p, p')$  is the space of functions  $u(x, t)$  of the form

$$u(x, t) = D^{2r}v(x, t), \quad v(x, t) \in \mathcal{L}_p(0, T; \dot{W}_p^{(k)}), \quad u'(x, t) \in \mathcal{L}_{p'}(0, T; \dot{W}_{p'}^{(-k)}),$$

where  $k \geq l$ ,  $2r = k - l$ .

Finally, we note that constants will be denoted by the letter  $K$ .

## § 1. Nonlinear strongly parabolic equations.

The case  $k \leq l$ . Put  $l - k = 2r$ , and in the rectangle  $Q$  consider the problem

$$\mathcal{L}(u) \equiv (-1)^r u' + D^k A(x, t, D^l u) = h(x, t); \quad (1)$$

$$u(x, 0) = 0, \quad u|_{\Gamma} = 0, \dots, \quad D^{r-1} u|_{\Gamma} = 0, \quad D^{2r} u|_{\Gamma} = 0, \dots, \quad D^{l-1} u|_{\Gamma} = 0 * . \quad (2)$$

**Assumptions.** I. The function  $A(x, t, \xi)$  is continuous in all arguments for  $(x, t) \in Q$  and arbitrary  $\xi$ , and

$$|A(x, t, \xi)| \leq K(|\xi|^{p-1} + 1),$$

where  $p > 1$  is a fixed number;

$$h(x, t) \in \mathcal{L}_{p'}(0, T; \dot{W}_p^{(-k)}).$$

II. **Parabolicity condition.** For every function

$$u(x, t) \in \mathcal{L}_p(0, T; W_p^{(l)})$$

and every  $t \in [0, T]$  the inequality

$$(-1)^k (A(x, t, D^l u), D^l u) \geq a_0 (\|D^l u\|^{p-1}, \|D^l u\|) - K(t)$$

holds, where  $a_0 > 0$ ,  $K(t)$  is a continuous function on  $[0, T]$ , and

$$(u, v) = \int_a^b uv \, dx.$$

III. **Definiteness condition** (strong parabolicity). For any  $u(x, t)$  and  $v(x, t)$  from  $\mathcal{L}_p(0, T; W_p^{(l)})$  the inequality

$$\begin{aligned} & (-1)^k [A(x, t, D^l u) - A(x, t, D^l v), D^l(u - v)] \geq \\ & \geq a_1 [\|D^l(u - v)\|^{p-1}, \|D^l(u - v)\|]; \\ & a_1 > 0, \quad [u, v] = \int_Q uv \, dx \, dt. \end{aligned}$$

\* The case of nonzero conditions that admit extension into  $Q$  from the space  $H(H_1)$  is treated analogously.

**Theorem 1.** *If conditions I-III are satisfied, then the mapping  $\mathcal{L} : H \rightarrow \mathcal{L}_{p'}(0, T; \dot{W}_p^{(-k)})$  is a homeomorphism.*

**Proof.** The most essential point in the proof of this theorem is the solvability of problem (1), (2). To prove this fact, choose a system of smooth functions  $v_1(x), v_2(x), \dots$ , complete in  $\dot{W}_p^{(k)}$ . We shall seek an approximate solution  $u_n(x, t)$  from the relation  $z_n(x, t) = D^{2r} u_n(x, t)$  under the condition

$$u_n|_{\Gamma} = 0, \dots, \quad D^{r-1}|_{\Gamma} = 0,$$

where

$$z_n = \sum_{\nu=1}^n c_{\nu n}(t) v_{\nu}(x).$$

The unknown functions  $c_{\nu n}(t)$  are determined from the system of ordinary differential equations

$$\begin{aligned} (-1)^r(u'_n, v_{\nu}) + (-1)^k(A(x, t, D^l u_n), D^k v_{\nu}), \quad c_{\nu n}(0) = 0 \\ (\nu = 1, \dots, n). \end{aligned} \quad (3)$$

**Lemma.** Let  $c = (c_1(t), \dots, c_n(t))$ , and let the vector function  $\mathbf{f}(t, c)$  be continuous in  $t$  and  $c$ . Suppose, further, that for any  $c$  and any  $t \in [0, T]$  the inequality

$$\int_0^t \mathbf{f}(\tau, c) c \, d\tau \geq - \left( a_2 \int_0^t |c|^2 \, d\tau + K(t) \right),$$

holds, where  $a_2 \geq 0$ ,  $K(t) \geq 0$  is a continuous function on  $[0, T]$ . Then the system

$$c' + \mathbf{f}(t, c) = 0, \quad c(0) = 0$$

has at least one solution defined on the whole interval  $[0, T]$ .

From this lemma (with a suitable choice of  $v_1(x), v_2(x), \dots$ ) there follows the solvability of system (3). For the approximate solutions  $u_n(x, t)$  the a priori estimate

$$[|D^l u_n|^{p-1}, |D^l u_n|] \leq K$$

is valid. From this estimate it follows that there exists a function  $u(x, t) \in \mathcal{L}(0, T; W_p^{(l)})$  and a subsequence (which we again denote by  $u_n(x, t)$ ) such that

$$u_n \rightarrow u, \dots, \quad D^l u_n \rightarrow D^l u$$

weakly in  $\mathcal{L}_p(Q)$ . Further, by virtue of I we may assume that

$$A(x, t, D^l u_n) \rightarrow a(x, t)$$

weakly in  $\mathcal{L}_{p'}(Q)$ . It is easy to see that the function  $u(x, t)$  satisfies the integral identity

$$(-1)^{r+1}[u, v'] + (-1)^k[a(x, t), D^k v] = [h, v], \quad \forall v \in \dot{W}_{p,2}^{(k,1)}. \quad (4)$$

Putting  $v = \psi(x)\varphi(t)$ , where  $\psi \in \dot{W}_p^{(k)}$ ,  $\varphi \in C^\infty(0, T)$ , and taking into account that  $a(x, t) \in \mathcal{L}_{p'}(Q)$ , we derive from (4) that

$$u' \in \mathcal{L}_{p'}(0, T; \dot{W}_p^{(-k)}),$$

i.e.  $u \in H$ . Hence it follows that  $u(x, t)$  satisfies the relation

$$(-1)^r[u', v] + (-1)^k[a(x, t), D^k v] = [h, u], \quad \forall v \in \mathcal{L}_p(0, T; \dot{W}_p^{(k)}).$$

Comparing these relations, we see that  $u(x, 0) = 0$ .

It remains to prove that

$$D^k a(x, t) = D^k A(x, t, D^l u).$$

For this purpose set

$$W_N = \{w(x, t), |w|_\Gamma = 0, \dots, D^{r-1}w|_\Gamma = 0, D^{2r}w = \sum_{\nu=1}^N \alpha_\nu(t)v_\nu(x)\},$$

where  $\alpha_\nu(t) \in C^\infty(0, T)$ ,  $N \geq 1$  is an integer, and use condition III, according to which for any function  $w(x, t) \in W_N$  we have

$$\begin{aligned} & (-1)^r [u'_n - w', D^{2r}(u_n - w)] + \\ & + (-1)^k [A(x, t, D^l u_n) - A(x, t, D^l w), D^l(u_n - w)] \geq 0. \end{aligned} \quad (5)$$

Now note that from (3) it obviously follows that  $u_n(x, t)$  satisfies the relation

$$(-1)^r [u'_n, v] + (-1)^k [A(x, t, D^l u_n), D^k v] = [h, v], \quad \forall v = \sum_{\nu=1}^n \alpha_\nu(t)v_\nu(x).$$

In particular, for  $w \in W_N$  with  $N \leq n$  we obtain

$$\begin{aligned} & (-1)^r [u'_n, D^{2r}(u_n - w)] + (-1)^k [A(x, t, D^l u_n), D^l(u_n - w)] \\ & = [h, D^{2r}(u_n - w)], \end{aligned}$$

which together with (5) gives the inequality

$$\begin{aligned} & [h, D^{2r}(u_n - w)] + (-1)^{r+1} [w', D^{2r}(u_n - w)] + \\ & + (-1)^{k+1} [A(x, t, D^l w), D^l(u_n - w)] \geq 0. \end{aligned} \quad (6)$$

Since (6) contains no nonlinearities in  $u_n$ , as  $n \rightarrow \infty$  we obtain

$$\begin{aligned} & [h, D^{2r}(u - w)] + (-1)^{r+1} (w', D^{2r}(u - w)) + \\ & + (-1)^{k+1} [A(x, t, D^l w), D^l(u - w)] \geq 0. \end{aligned}$$

Further, since  $D^{2r}(u - w) \in \mathcal{L}_p(0, T; \dot{W}_p^{(k)})$ , then, taking (5) into account, we find

$$(-1)^r [u' - w', D^{2r}(u - w)] + + (-1)^k [a(x, t) - A(x, t, D^l w), D^l(u - w)] \geq 0.$$

The last inequality has been proved for  $w \in W_N$ ; but since the linear combinations  $\sum_{\nu=1}^N \alpha_\nu(t)v_\nu(x)$  are strongly dense in  $\mathcal{L}_p(0, T; \dot{W}_p^{(k)})$ , it is valid for any

function  $w(x, t)$  ( $w|_{\Gamma} = 0, \dots, D^{r-1}w|_{\Gamma} = 0$ ), for which  $D^{2r}w \in \mathcal{L}_p(0, T; \dot{W}_p^{(k)})$ . In particular, putting  $w = u - \xi\zeta$ ,  $\zeta \in H$ ,  $\xi \rightarrow +0$ , we obtain that

$$(-1)^k[a(x, t) - A(x, t, D^l u), D^l \zeta] \geq 0. \quad (7)$$

The latter is possible only in the case when  $D^k a(x, t) = D^k A(x, t, D^l u)$ .<sup>\*</sup> The uniqueness of the solution follows from condition III, and from this also follows the continuity of the inverse operator  $\mathcal{L}^{-1}$ . The theorem is proved.

**Example.**  $\mathcal{L}(u) \equiv (-1)^r u' + (-1)^k D^k(|D^l u|^{p-1} \text{sign } D^l u)$ ; all the conditions of Theorem 1 for the operator  $\mathcal{L}(u)$  are fulfilled. Let us note the special case  $k = 0$ ,  $l = 2m$ . In this case

$$\mathcal{L}(u) \equiv (-1)^m u' + |D^{2m} u|^{p-1} \text{sign } D^{2m} u$$

under the conditions of the first boundary-value problem maps  $\dot{W}_{p,p}^{(2m,1)}$  homeomorphically onto  $\mathcal{L}_p$ .

**Theorem 2.** *If  $k \geq l$  and conditions I-III are fulfilled, then the mapping*

$$\mathcal{L} : H_1 \rightarrow \mathcal{L}'_p(0, T; \dot{W}_p^{(-k)})$$

*is a homeomorphism.*

## § 2. General nonlinear parabolic equations.

The case  $k_0 \leq l_0$ . Let  $(k, l, s, n)$  be some collection of sets of nonnegative integers, where  $k + l = 2m$ ,  $s + n \leq 2m - 1$ ,  $n \leq l_0 - 1$ ,  $l_0 = \max l$ ;  $k_0 = 2m - l_0$ ,  $l_0 - k_0 = 2r_0$ . For a given collection of sets we study the problem

$$\begin{aligned} (-1)^{r_0} u' + \mathcal{L}(u) + T_1(u) + T_2(u) &\equiv (-1)^{r_0} u' + \sum_{k+l=2m} D^k A_k(x, t, u, \dots, D^l u) + \\ &+ \sum_{k+l \leq 2m-1, s \geq k_0} D^s B_s(x, t, u, \dots, D^n u) + \\ &+ \sum_{k+l \leq 2m-1, s < k_0} D^s B_s(x, t, u, \dots, D^n u) = 0, \end{aligned} \quad (8)$$

$$u(x, 0) = 0, \quad u|_{\Gamma} = 0, \dots, D^{r_0-1} u|_{\Gamma} = 0, \quad D^{2r_0} u|_{\Gamma} = 0, \dots, D^{l_0-1} u|_{\Gamma} = 0. \quad (9)$$

**Assumptions.**  $I'$ . The functions  $A_k(x, t, \xi^0, \dots, \xi^l)$ ,  $B_s(\dots)$  are continuous in all arguments, and

$$|A_k(x, t, \xi^0, \dots, \xi^l)| \leq K \prod_{i=1}^l (|\xi^i|^{p_i} + 1), \quad |B_s(x, t, \xi^0, \dots, \xi^n)| \leq$$

$$\leq K \prod_{i=1}^n (|\xi^i|^{q_i} + 1),$$

where  $p_i \geq 0$ ,  $q_i \geq 0$ ,  $\sum p_i \leq p-1$ ,  $\sum q_i \leq p-1$ . In addition, suppose that for  $k > k_0$  ( $s > k_0$ ) the functions  $A_k(x, t, \xi^0, \dots, \xi^l)$  ( $B_s(\dots)$ ) are differentiable with respect to  $x, t$  and  $\vec{\xi} k - k_0$  ( $s - k_0$ ) times, and each differentiation with respect to  $x$  and  $t$  does not increase the growth in  $\xi$ , while each differentiation with respect to  $\xi^i$  ( $i = 1, \dots, n; l$ ) decreases the growth in  $\xi^i$  by one. We introduce the notation:

$$\mathcal{L}(u, v) \equiv (-1)^{k_0} \sum_{k+l=2m} [D^{k-k_0} A(x, t, u, \dots, D^l u), D^{k_0} v],$$

and  $T_1(u, v)$  and  $T_2(u, v)$  are defined analogously.

**II'. Parabolicity condition.** For any function

$$u(x, t) \in \mathcal{L}_p(0, T; W_p^{(l_0)})$$

$$\begin{aligned} \mathcal{L}_0(u, D^{2r_0} u) &\equiv \mathcal{L}(u, D^{2r_0} u) + \\ + T_1(u, D^{2r_0} u) + T_2(u, D^{2r_0} u) &\geq a_0 [|D^{l_0} u|^{p-1}, |D^{l_0} u|] - K; \quad a_0 > 0. \end{aligned}$$

\* Let us note that the idea of monotonicity was used earlier by a number of authors: M. M. Vainberg and R. I. Kachurovskii, J. Minty, F. Browder, and others.

**III'. The condition of semiboundedness of the "variation."** For any  $u \in \mathcal{L}_p(0, T; W_p^{(l_0)})$  and  $v \in \mathcal{L}_p(0, T; W_p^{(l_0)})$  from the ball  $\|u\|_{l_0, p} \leq R$ , the inequality

$$\mathcal{L}_0(u, D^{2r_0}(u-v)) - \mathcal{L}_0(v, D^{2r_0}(u-v)) \geq -c(R, \|u-v\|_{l_0-1, p})$$

holds, where  $c(R, \rho) \geq 0$  is a continuous function such that, for any  $R$  and  $\rho$ ,  $c(R, \xi\rho)/\xi \rightarrow 0$  as  $\xi \rightarrow +0$ .

**Theorem 3.** *If conditions I–III' are satisfied, then problem (8), (9) has at least one solution in the space*

$$H = H(l_0, k_0, r_0, p, p').$$

**Proof.** First we shall have to solve an elliptic equation containing a small parameter  $\varepsilon > 0$ , namely the equation

$$(-1)^{r_0+1} \varepsilon u'' + (-1)^r u' + \mathcal{L}(u) + T_1(u) + T_2(u) = 0 \quad (10)$$

with the additional condition  $u'(x, T) = 0$ . We seek an approximate solution of this problem from the relation  $D^{2r_0} u_{n\varepsilon} = z_{n\varepsilon}$ , under the conditions  $u_{n\varepsilon}|_{\Gamma} = 0, \dots,$

$$\dots, \quad D^{r_0-1} u_{n\varepsilon}|_{\Gamma} = 0,$$

where

$$z_{n\varepsilon}(x, t) = \sum_{\nu=1}^n c_{\nu n} v_{\nu}(x, t);$$

$v_1(x, t), v_2(x, t), \dots$  is a system of smooth functions, complete in  $\dot{W}_{p,2}^{(l_0,1)}$ . The unknown constants  $c_{\nu n}$  are determined from the system of equations

$$\begin{aligned} (-1)^{r_0} \varepsilon [u'_{n\varepsilon}, v'_{\nu}] + (-1)^{r_0} [u'_{n\varepsilon}, v_{\nu}] + \mathcal{L}(u_{n\varepsilon}, v_{\nu}) + T_1(u_{n\varepsilon}, v_{\nu}) + T_2(u_{n\varepsilon}, v_{\nu}) &= 0, \\ \nu &= 1, \dots, n. \end{aligned}$$

The solvability of this system follows from the lemma of M. I. Vishik (1); moreover, the following a priori estimate is valid:

$$\varepsilon [D^{r_0} u'_{n\varepsilon}, D^{r_0} u'_{n\varepsilon}] + [|D^{l_0} u_{n\varepsilon}|^{p-1}, |D^{l_0} u_{n\varepsilon}|] \leq K.$$

From this estimate and J. Aubin's theorem (2) it follows that, for fixed  $\varepsilon > 0$ , there exists a function  $u_{\varepsilon}(x, t) \in \mathcal{L}_p(0, T; W_p^{(l_0)})$  and a subsequence (denote it again by  $u_{n\varepsilon}(x, t)$ ) such that  $u_{n\varepsilon} \rightarrow u_{\varepsilon}$  ( $n \rightarrow \infty$ ) strongly in  $\mathcal{L}_p(0, T; W_p^{(l_0-1)})$ , and, moreover,  $D^{l_0} u_{n\varepsilon} \rightarrow D^{l_0} u_{\varepsilon}$  weakly in  $\mathcal{L}_p(Q)$ , while  $D^{r_0} u'_{n\varepsilon} \rightarrow D^{r_0} u'_{\varepsilon}$  weakly in  $\mathcal{L}_2(Q)$ . As in § 1, we conclude that

$$\begin{aligned} (-1)^{r_0} \varepsilon [u'_{\varepsilon}, v'] + (-1)^{r_0} [u'_{\varepsilon}, v] + \mathcal{L}(u_{\varepsilon}, v) + T_1(u_{\varepsilon}, v) + T_2(u_{\varepsilon}, v) &= 0, \\ \forall v \in \dot{W}_{p,2}^{(k_0,1)}. \end{aligned}$$

Hence, putting  $v = \psi(x)\varphi(t)$ ,  $\psi \in \dot{W}_p^{(k_0)}$ ,  $\varphi \in \dot{C}^{\infty}(0, T)$ , we find that

$$-\varepsilon u''_{\varepsilon} + u'_{\varepsilon} = g_{\varepsilon}(t),$$

where

$$g_{\varepsilon}(t) \in \mathcal{L}_{p'}(0, T; \dot{W}_{p'}^{(-k_0)})$$

and ranges there over a bounded set. Solving this equation under the condition  $u'_{\varepsilon}(T) = 0$ , we obtain that

$$u'_{\varepsilon} \in \mathcal{L}_{p'}(0, T; \dot{W}_{p'}^{(-k_0)}) \quad \text{and} \quad \|u'_{\varepsilon}\|_{-k_0, p'} \leq K$$

as  $\varepsilon \rightarrow 0$ . Therefore, again by J. Aubin's theorem, one may assume that

$$u_{\varepsilon} \rightarrow u \in H$$

strongly in

$$\mathcal{L}_p(0, T; W_p^{(l-1)}),$$

and

$$D^{l_0} u_{\varepsilon} \rightarrow D^{l_0} u$$

weakly in  $\mathcal{L}_p(Q)$ . We then conclude (again repeating the scheme of § 1) that  $u(x, t)$  is the desired solution of problem (8), (9). The theorem is proved.

**Theorem 4.** *If  $k_0 \geq l_0$  and conditions I–III' are satisfied, then equation (8) has at least one solution in the space*

$$H_1 = H_1(l_0, k_0, r_0, p, p').$$

§ 3. It is obvious that what was said in §§ 1 and 2 generalizes to the cylinder

$$Q = G \times [0, T], \quad G \subset R^n,$$

if, formally, in equations (1), (8) one replaces  $D$  by a suitable elliptic operator  $A$ . For example, the results of Theorem 1 are valid for the problem

$$\begin{aligned} (-1)^m u' + |\Delta^m u|^{p-1} \operatorname{sign} \Delta^m u &= h(x, t), \\ u(x, 0) = 0, \quad D^\omega u|_\Gamma &= 0, \quad \omega = (\omega_1, \dots, \omega_n), \\ |\omega| = \omega_1 + \dots + \omega_n &\leq m - 1. \end{aligned}$$

In conclusion, I take this opportunity to express my gratitude to Prof. M. I. Vishik for his attention to my work.

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Received  
4 I 1965

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