MATHEMATICS

1965

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Abstract

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MATHEMATICS

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ON A CLASS OF INTERPOLATION POLYNOMIALS WITH UNFIXED NODES

(Presented by Academician S. N. Bernstein on 2 XII 1964)

In the work of Ya. Mycielski and S. Paszkowski ([1]) the following interesting theorem was proved. Let $v_0, v_1, \dots, v_n$ be given positive numbers and let $[a, b]$ be a real interval. There exists a unique sequence of points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and such an algebraic polynomial $P_n(x)$ of degree n that

$$P_n(x_k) = (-1)^{n-k} v_k \quad (k = 0, 1, \dots, n);$$

$$P_n'(x_i) = 0 \quad (i = 1, \dots, n-1).$$

In particular, if all $v_k = 1$, then $P_n(x)$ is the Chebyshev polynomial. Here we generalize this theorem and give a proof based on an entirely different idea than in ([1]). Consider a sequence of functions $\{\varphi_k(x)\}_{k=0}^{n}$, continuously differentiable on $[a, b]$. We shall write $\{\varphi_k(x)\} \in T_n$ if every polynomial

$$P(x) = \sum_{k=0}^{n} a_k \varphi_k(x) \quad \left(\sum_{k=0}^{n} |a_k| > 0 \right)$$

on $[a, b]$ has $\le n$ zeros. If, moreover, its derivative $P'(x)$ also has $\le n$ zeros on $[a, b]$, then we shall write $\{\varphi_k(x)\} \in T_n'$.

Fix positive numbers $v_0, v_1, \dots, v_n$ and, having chosen a system of interpolation nodes

$$a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b,$$

construct, with respect to $\{\varphi_k(x)\}$, the polynomial

$$P(x) = P(x; \xi_0, \xi_1, \dots, \xi_n) = \sum_{k=0}^{n} (-1)^{n-k} v_k l_k(x), \tag{1}$$

where

$$l_k(x) = D_k(x) / D[\varphi_0(\xi_0), \varphi_1(\xi_1), \dots, \varphi_n(\xi_n)],$$

$$D[\varphi_0(\xi_0), \varphi_1(\xi_1), \dots, \varphi_n(\xi_n)] = \begin{vmatrix} \varphi_0(\xi_0) & \varphi_1(\xi_0) & \cdots & \varphi_n(\xi_0) \\ \varphi_0(\xi_1) & \varphi_1(\xi_1) & \cdots & \varphi_n(\xi_1) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_0(\xi_n) & \varphi_1(\xi_n) & \cdots & \varphi_n(\xi_n) \end{vmatrix},$$

$$D_k(x) = D[\varphi_0(\xi_0), \dots, \varphi_{k-1}(\xi_{k-1}), \varphi_k(x), \varphi_{k+1}(\xi_{k+1}), \dots, \varphi_n(\xi_n)].$$

The class of polynomials of the form (1), for fixed $v_k > 0$ and arbitrary $\xi_1 < \dots < \xi_{n-1}$ from (a, b), will be denoted by V_n.

Theorem 1. *Let $\{\varphi_k(x)\} \in T_n$, and suppose that the points*

$$a = \xi_0 < \dots < \xi_{i-1} < \xi_{i+1} < \dots < \xi_n = b \quad (1 \le i \le n-1)$$

and the numbers $w_0, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_n$ are given such that

$$w_{i-1} > 0, \quad w_i \le 0, \quad w_{i+1} > 0.$$

In (ξ_{i-1}, ξ_{i+1}) there exists such a point ξ_i and such a polynomial $L(x)$ with respect to the system $\{\varphi_k(x)\}$ that

$$L(\xi_k) = w_k \quad (k = 0, 1, \dots, n); \qquad L'(\xi_i) = 0.$$

Adjoin to the points ξ_k $(k \ne i)$, as an interpolation node, an arbitrary point z from (ξ_{i-1}, ξ_{i+1}), and construct the polynomial

$$L(x) = L(x; \xi_0, \dots, \xi_{i-1}, z, \xi_{i+1}, \dots, \xi_n) = \sum_{k=0}^{n} w_k l_k(x), \tag{2}$$

where $l_k(x)$ are the fundamental polynomials for the system of nodes $\xi_0 < \dots < \xi_{i-1} < z < \xi_{i+1} < \dots < \xi_n$. Differentiating formula (2) with respect to x at the point $x = z$, we obtain

$$L'(z) D_i(z) = \sum_{k=0}^{n} w_{kD} k'(z) = \Phi(z), \quad \text{where} \quad D'_k(z) = D'_k(x)\big|_{x=z}.$$

We shall show that $\Phi(\xi_{i-1})\Phi(\xi_{i+1}) < 0$. Consequently, there is a point $z = \xi_i$, $\xi_{i-1} < \xi_i < \xi_{i+1}$, such that $\Phi(\xi_i) = 0$, and hence also $L'(\xi_i) = 0$, since $D_i(\xi_i) = D[\varphi_0(\xi_0), \dots, \varphi_n(\xi_n)] \ne 0$ for $\{\varphi_k(x)\} \in T_n$. If $i \ne k$, then $D'_k(z) = D[\varphi_0(\xi_0), \dots, \varphi_i(z), \dots, \varphi'_k(z), \dots, \varphi_n(\xi_n)]$. For $i \ne k$ the determinant $D'_k(\xi_j) = 0$, if $j \ne i, k$, since it contains two identical rows. Thus,

$$\Phi(\xi_{i-1}) = w_{i-1} D'_{i-1}(\xi_{i-1}) + w_{iD} i'(\xi_{i-1}), \quad \Phi(\xi_{i+1}) = w_{iD} i'(\xi_{i+1}) + w_{i+1} D'_{i+1}(\xi_{i+1})$$

The polynomial $D_i(x)$ vanishes at the n points ξ_j $(j \ne i)$; hence the ξ_j are its simple zeros, and therefore, in particular, $D'_i(\xi_{i-1}) \ne 0$. Since the determinant

$D'_{i-1}(\xi_{i-1})$ is obtained from the determinant $D'_i(\xi_{i-1})$ by interchanging two of its rows, we have $D'_{i-1}(\xi_{i-1}) = -D'_i(\xi_{i-1})$. We obtain

$$\Phi(\xi_{i-1}) = (-w_{i-1} + w_i)D'_i(\xi_{i-1}), \quad \Phi(\xi_{i+1}) = (w_i - w_{i+1})D'_i(\xi_{i+1}),$$

and since ξ_{i-1}, ξ_{i+1} are neighboring simple zeros of the polynomial $D_i(x)$, it follows that $D'_i(\xi_{i-1})D'_i(\xi_{i+1}) < 0$, and consequently $\Phi(\xi_{i-1})\Phi(\xi_{i+1}) < 0$.

Theorem 2. Let $\{\varphi_k(x)\} \in T'_n$, and let points be given

$$a = \xi_0 < \xi_1 < \cdots < \xi_{i-1} < \xi_{i+r} < \cdots < \xi_{n-1} < \xi_n = b$$
$$(1 \le i \le n-1; \quad 1 \le r \le n-1; \quad i + r \le n).$$

In (ξ_{i-1}, ξ_{i+r}) there exists a unique system of r points $\xi_i < \xi_{i+1} < \cdots < \xi_{i+r-1}$ such that $P(x) = P(x; \xi_0, \ldots, \xi_n) \in V_n$ satisfies the conditions

$$P'(\xi_i) = P'(\xi_{i+1}) = \cdots = P'(\xi_{i+r-1}) = 0. \tag{3}$$

Theorem 2 generalizes the result of [1] in two directions: algebraic polynomials are replaced by polynomials with respect to $\{\varphi_k(x)\} \in T'_n$, and instead of $r = n-1$ the condition $1 \le r \le n-1$ is introduced.

We shall prove Theorem 2 by induction applied to the number r. For $r = 1$ the result follows from Theorem 1. Indeed, suppose that in (ξ_{i-1}, ξ_{i+1}) there exist two points ξ_{i1} and ξ_{i2} such that the corresponding polynomials $P_1(x) \in V_n$ and $P_2(x) \in V_n$ satisfy the conditions $P'_1(\xi_{i1}) = 0$, $P'_2(\xi_{i2}) = 0$. Since ξ_{i1} and ξ_{i2} are the unique points of maximum of the polynomials $(-1)^{n-i}P_1(x)$, $(-1)^{n-i}P_2(x)$ in (ξ_{i-1}, ξ_{i+1}), it follows that $v_i = |P_1(\xi_{i1})| > (-1)^{n-i}P_2(\xi_{i1})$, $(-1)^{n-i}P_1(\xi_{i2}) < |P_2(\xi_{i2})| = v_i$. Hence it follows that the polynomial $P_1(x) - P_2(x)$ has a zero in (ξ_{i1}, ξ_{i2}). On the other hand, $P_1(\xi_k) - P_2(\xi_k) = 0$ $(k \ne i)$; consequently, the total number of zeros of $P_1(x) - P_2(x)$ is greater than n on $[a, b]$.

We shall now assume that Theorem 2 has already been established for some fixed number r $(1 \le r \le n-2)$ for any i, and shall show that in this case it is valid also for the number $r + 1$. Thus, let points be given $a = \xi_0 < \xi_1 < \cdots < \xi_{i-1} < \xi_{i+r+1} < \cdots < \xi_n = b$. Adjoin to them an arbitrary point ξ'_{i+r}, $\xi_{i-1} < \xi'_{i+r} < \xi_{i+r+1}$. By the induction hypothesis, in (ξ_{i-1}, ξ'_{i+r}) there are uniquely found r points $\xi'_i < \xi'_{i+1} < \cdots < \xi'_{i+r-1}$ such that the polynomial

$$P_1(x) = P_1(x; \xi_0, \ldots, \xi_{i-1}, \xi'_i, \ldots, \xi'_{i+r}, \xi_{i+r+1}, \ldots, \xi_n) \in V_n$$

satisfies (3) at these points. If $P'_1(\xi'_{i+r}) = 0$, then $P_1(x)$ is the desired polynomial. If, however, $P'_1(\xi'_{i+r}) \ne 0$, then in $(\xi'_{i+r-1}, \xi_{i+r+1})$ there is a point η'_{i+r}, $\eta'_{i+r} \ne \xi'_{i+r}$, such that $P_1(\eta'_{i+r}) = (-1)^{n-i-r}v_{i+r}$. Next choose

$$\xi''_{i+r} = \frac{1}{2}(\xi'_{i+r} + \eta'_{i+r})$$

and repeat the preceding reasoning. In

(ξ_{i-1}, ξ''_{i+r}) there are uniquely determined r points $\xi''_i < \dots < \xi''_{i+r-1}$ such that the polynomial

$$P_2(x) = P_2(x; \xi_0, \dots, \xi_{i-1}, \xi''_i, \dots, \xi''_{i+r}, \xi_{i+r+1}, \dots, \xi_n) \in V_n$$

satisfies conditions (3) at these points. Suppose that $P'_2(\xi''_{i+r}) \neq 0$; denote by $x'_1 < \dots < x'_n$ the zeros of $P_1(x)$, and let us show that $P_1(x) - P_2(x) \neq 0$ in (x'_{i+r}, x'_{i+r+1}). Indeed, if at some point z, $x'_{i+r} < z < x'_{i+r+1}$,

$$P_1(z) = P_2(z) = (-1)^{n-i-r} v^*_{i+r}, \qquad v^*_{i+r} > 0,$$

then this would mean that the two distinct polynomials

$$P_1(x) = P_1(x; \xi_0, \dots, \xi'_{i+r-1}, z, \xi_{i+r+1}, \dots, \xi_n) \in V^*_n,$$

$$P_2(x) = P_2(x; \xi_0, \dots, \xi''_{i+r-1}, z, \xi_{i+r+1}, \dots, \xi_n) \in V^*_n$$

satisfy conditions (3), where V^*_n is the class of polynomials obtained from V_n by replacing v_{i+r} by v^*_{i+r}. But this contradicts the induction hypothesis on the uniqueness of such a polynomial. Since $P_1(x) - P_2(x) \neq 0$ in $(\eta'_{i+r}, \xi'_{i+r}) \subset (x'_{i+r}, x'_{i+r+1})$, and since $P_1(\xi'_{i+r}) = P_1(\eta'_{i+r}) = P_2(\xi'_{i+r})$, it follows that $P_2(x) < |P_1(x)|$ for $x \in [\eta'_{i+r}, \xi'_{i+r}]$, and in $(\eta'_{i+r}, \xi'_{i+r})$ there is a point $\eta''_{i+r} \neq \xi'_{i+r}$ such that

$$P_2(\eta''_{i+r}) = (-1)^{n-i-r} v_{i+r}.$$

It is clear that $|\eta''_{i+r} - \xi''_{i+r}| < \frac{1}{2}|\eta'_{i+r} - \xi'_{i+r}|$. We shall now show that

$$\xi_i < \xi''_i, \dots, \xi'_{i+r-1} < \xi''_{i+r-1}. \tag{4}$$

We shall regard the polynomial $P_2(x)$ as a continuous image of the polynomial $P_1(x)$ corresponding to the continuous displacement of the point ξ'_{i+r} to ξ''_{i+r}. Move the point ξ'_{i+r} to the point $\tilde{\xi}_{i+r}$, and assume that $|\xi'_{i+r} - \tilde{\xi}_{i+r}|$ is as small as we need. In $(\xi_{i-1}, \tilde{\xi}_{i+r})$ there are uniquely determined r points $\tilde{\xi}_i < \dots < \tilde{\xi}_{i+r-1}$, which are, respectively, small displacements of the points $\xi'_i < \dots < \xi'_{i+r-1}$, such that the polynomial

$$Q'(x) = Q'(x; \xi_0, \dots, \xi_{i-1}, \tilde{\xi}_i, \dots, \tilde{\xi}_{i+r}, \dots, \xi_n) \in V_n'$$

satisfies conditions (3) at the points $\tilde{\xi}_i, \dots, \tilde{\xi}_{i+r-1}$. If $|\tilde{\xi}_k - \xi_k'|$ $(k = i, \dots, i+r-1)$ are sufficiently small, then

$$\xi_i < \tilde{\xi}_i, \dots, \xi_{i+r-1}' < \tilde{\xi}_{i+r-1}. \tag{5}$$

Indeed, if for some k we had $\tilde{\xi}_k < \xi_k'$, then the polynomial $P_1(x) - Q'(x)$ would have two zeros in a neighborhood of the point ξ_k'. In a neighborhood of each of the ξ_j' it has at least one zero, so that the total number of its zeros in (ξ_{i-1}, ξ_{i+r+1}) would be $\geq r+1$, while at the remaining $n-r$ nodes ξ_s, $P_1(\xi_s) - Q'(\xi_s) = 0$; hence on $[a, b]$ the function $P_1(x) - Q'(x)$ would have $\geq n+1$ zeros. This proves also inequalities (4), since inequalities (5) cannot be violated as $\xi_{i+r}' \to \xi_{i+r}''$.

Continuing the construction process by the same scheme, we obtain $\{P_m(x)\} \subset V_n$. We see that $\{\xi_k^{(m)}\}$ $(k = i, \dots, i+r-1)$, for fixed k, is increasing and bounded; consequently, as $m \to \infty$ it has a limit ξ_k; $\{\xi_{i+r}^{(m)}\}$ has a limit as $m \to \infty$, since

$$[\eta_{i+r}^{(m)}, \xi_{i+r}^{(m)}] \subset [\eta_{i+r}^{(m-1)}, \xi_{i+r}^{(m-1)}]$$

and

$$|\eta_{i+r}^{(m)} - \xi_{i+r}^{(m)}| < 2^{-m+1} |\eta_{i+r}' - \xi_{i+r}'|.$$

Let us note that $\xi_k \neq \xi_{k+1}$. Indeed, if $\xi_k = \xi_{k+1}$, then $\{P_m(x)\}$ would converge to a discontinuous function; meanwhile, $\{P_m(x)\}$ is uniformly bounded on $[\xi_{i+r-1}', \xi_{i+r}']$, hence it is compact and can have as its limit only a polynomial from T_n'. Put

$$P(x) = \lim_{m \to \infty} P_m(x).$$

It is easy to see that $P(x) = P(x; \xi_0, \dots, \xi_n) \in V_n$ and that

$$P'(\xi_i) = \dots = P'(\xi_{i+r}) = 0. \tag{6}$$

We shall now show that the points $\xi_i < \dots < \xi_{i+r}$ and the corresponding polynomial $P(x) \in V_n$ are uniquely determined by conditions (6). Suppose that there exists another system of points $\zeta_i < \dots < \zeta_{i+r}$ such that the polynomial $Q(x) \in V_n$ corresponding to it satisfies conditions (6) at these points. If $\xi_{i+r} = \zeta_{i+r}$, then for the two systems of r points $\xi_i < \dots < \xi_{i+r-1}$ and $\zeta_i < \dots < \zeta_{i+r-1}$, and for the polynomials $P(x)$ and $Q(x)$, conditions (3) would

be fulfilled, which contradicts the inductive assumption of uniqueness. Let, for definiteness, $\xi_{i+r} < \zeta_{i+r}$. We begin our construction process by choosing in (ξ_{i-1}, ξ_{i+r+1}) an arbitrary point ξ'_{i+r}. Denote by X the upper bound of those ξ'_{i+r} starting from which we arrive at the points $\xi_i < \dots < \xi_{i+r}$ and at the polynomial $P(x)$. It is clear that $X \leq \zeta_{i+r}$, since the sequence of contracting intervals $[\eta^{(m)}_{i+r}, \xi^{(m)}_{i+r}]$ is constructed uniquely after the point ξ'_{i+r} has been chosen. Suppose that $X = \zeta_{i+r}$. Choose $\xi'_{i+r} < X$ and so close to X that the polynomial $Q(x)$ does not change sign in (ξ'_{i+r}, X) and so that $\xi_{i+r} < \xi'_{i+r}$. Then ξ'_{i+r} will be the right endpoint of the interval $[\eta'_{i+r}, \xi'_{i+r}]$, and the corresponding polynomial $P_1(x)$ will be monotone on $[\xi'_{i+r}, \xi_{i+r+1}]$. Hence it follows that $|P_1(X)| < |Q(X)|$, $|P_1(\xi'_{i+r})| > |Q(\xi'_{i+r})|$. Consequently, in (ξ'_{i+r}, X) there is a point μ_{i+r} such that
$P_1(\mu_{i+r}) = Q(\mu_{i+r}) = (-1)^{n-i-r} v^*_{i+r}$, $v^*_{i+r} > 0$, and this means that two distinct polynomials

$$P_1(x) = P_1(x; \xi_0, \dots, \xi_{i+r-1}, \mu_{i+r}, \dots, \xi_n) \in V^*_n,$$
$$Q(x) = Q(x; \xi_0, \dots, \xi_{i+r-1}, \mu_{i+r}, \dots, \xi_n) \in V^*_n$$

satisfy conditions (3) at the points $\xi_i < \dots < \xi_{i+r-1}$ and $\zeta_i < \dots < \zeta_{i+r-1}$, respectively; V^*_n is the class obtained from V_n by replacing v_{i+r} by v^*_{i+r}. We have arrived at a contradiction with the inductive assumption of the uniqueness of such a polynomial. The case $X < \zeta_{i+r}$ is considered analogously. It is only necessary, in addition to the point $\xi'_{i+r} < X$, to choose another point $\zeta'_{i+r} > X$ and, with the aid of our process, to construct the polynomial $Q_1(x)$, and then repeat arguments similar to those given above.

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Received
1 XII 1964

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Note: Figure translations are in progress. See original paper for figures.