THEORY OF SHALLOW ANISOTROPIC PLASTIC SHELLS

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Abstract

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THEORY OF ELASTICITY

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THEORY OF SHALLOW ANISOTROPIC PLASTIC SHELLS

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1. The theory set forth pursues two aims: the determination of the load-carrying capacity of shallow orthotropic shells and the search for their rational structural forms.

The theory is based on assumptions customary for elastic shells ([1–3]) and on an approximate rigid-plastic computational scheme, according to which both elastic and elastoplastic parts of the shell are taken as rigid. In other words, the onset of the plastic state through the thickness of the shell is assumed to be instantaneous ([4,5]).

If the principal axes of anisotropy coincide with the axes of a rectangular coordinate system whose x and y axes lie in the horizontal plane of the shell plan, and whose z axis is directed vertically upward, then the set of initial equations takes the following form:

a) equilibrium equations ([2]):

$$\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial y} + P_x = 0, \qquad \frac{\partial T_2}{\partial y} + \frac{\partial S}{\partial x} + P_y = 0,$$
$$k_1 T_1 + k_2 T_2 + \frac{\partial^2 M_1}{\partial x^2} - 2\frac{\partial^2 H}{\partial x\,\partial y} + \frac{\partial^2 M_2}{\partial y^2} - P_z = 0; \tag{1}$$

b) geometrical equations ([2]):

$$\varepsilon_1 = \frac{\partial u}{\partial x} + k_1 w, \qquad \varepsilon_2 = \frac{\partial v}{\partial y} + k_2 w, \qquad \omega = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$
$$\chi_1 = -\frac{\partial^2 w}{\partial x^2}, \qquad \chi_2 = -\frac{\partial^2 w}{\partial y^2}, \qquad \chi = -\frac{\partial^2 w}{\partial x\,\partial y}; \tag{2}$$

c) physical equations ([4,5]):

$$F(T_1, T_2, S, M_1, M_2, H) \equiv$$

$$\equiv \frac{1}{h^2}\left(\frac{T_1^2}{\sigma_{s1}^2} - \frac{T_1T_2}{\sigma_{s1}\sigma_{s2}} + \frac{T_2^2}{\sigma_{s2}^2} + \frac{S^2}{\tau_s^2}\right) + \frac{12}{h^4}\left(\frac{M_1^2}{\sigma_{s1}^2} - \frac{M_1M_2}{\sigma_{s1}\sigma_{s2}} + \frac{M_2^2}{\sigma_{s2}^2} + \frac{H^2}{\tau_s^2}\right) = 1, \tag{3}$$

$$\frac{\varepsilon_1}{\partial F/\partial T_1} = \frac{\varepsilon_2}{\partial F/\partial T_2} = \frac{\omega}{\partial F/\partial S} = \frac{\chi_1}{\partial F/\partial M_1} = \frac{\chi_2}{\partial F/\partial M_2} = \frac{\chi}{\partial F/\partial H} = \lambda h, \tag{4}$$

$$\lambda = \left[\frac{1}{3}\left(\sigma_{s1}^2\varepsilon_1^2 + \sigma_{s1}\sigma_{s2}\varepsilon_1\varepsilon_2 + \sigma_{s2}^2\varepsilon_2^2\right) + \frac{\tau_s^2\omega^2}{4} + \right.$$
$$\left. + \frac{h^2}{36}\left(\sigma_{s1}^2\chi_1^2 + \sigma_{s1}\sigma_{s2}\chi_1\chi_2 + \sigma_{s2}^2\chi_2^2\right) + \frac{\tau_s^2h^2}{48}\chi^2\right]^{1/2}. \tag{5}$$

In these equations the following notation has been adopted: T_1, T_2 are normal forces; S is the shearing force; M_1, M_2 are bending moments; H is the twisting moment; k_1, k_2 are characteristic curvatures; P_x, P_y, P_z are the components of the intensity of the surface load along the coordinate axes; u, v, w are the velocities of displacement in the indicated directions; $\varepsilon_1, \varepsilon_2, \omega, \chi_1, \chi_2, \chi$ are the velocities of deformation and curvature of an element of the shell; h is its thickness; $\sigma_{s1}, \sigma_{s2}, \tau_s$ are the values of the yield limits in tension and shear; λ is the so-called "coefficient of plasticity," which is related to the rate of dissipation of mechanical energy D, referred to a unit area of the middle surface of the shell, by the relation

$$D = 2h\lambda. \tag{6}$$

Having at one' s disposal the yield condition (3) and the corresponding flow law (4), one can estimate the load-carrying capacity of the shell with the aid of the well-known theorems on the limit state. The formulation and proof of these theorems as applied to orthotropic shells are contained in ([5],[6]).

2. Let us show the application of the theory to the study of the load-carrying capacity of a shallow shell, hinged along its contour and subjected only to a uniform vertical load ($P_z = -P = -\text{const}$, $P_x = P_y = 0$). A kinematically possible velocity field of horizontal displacements may be taken in the form $u \equiv 0$, $v \equiv 0$, while the deflection velocity w may be taken in accordance with the bending of a flat plate supported along the contour. On the basis of formulas (6), (5), and (2) we may write:

$$D = 2h\left\{ \frac{1}{3}\left(\sigma_{s1}^2 k_1^2 + \sigma_{s1}\sigma_{s2}k_1k_2 + \sigma_{s2}^2k_2^2\right)w^2 + \right.$$

$$\left. + \frac{h^2}{36}\left[\sigma_{s1}^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \sigma_{s1}\sigma_{s2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \sigma_{s2}^2\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] + \frac{\tau_s^2 h^2}{48}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}^{1/2} <$$

$$< \frac{2h}{\sqrt{3}}\left(\sigma_{s1}^2 k_1^2 + \sigma_{s1}\sigma_{s2}k_1k_2 + \sigma_{s2}^2k_2^2\right)^{1/2} w +$$

$$+ \frac{h^2}{\sqrt{3}}\left\{\frac{1}{3}\left[\sigma_{s1}^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \sigma_{s1}\sigma_{s2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \sigma_{s2}^2\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] + \frac{\tau_s^2}{4}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}^{1/2}.$$

Hence, taking into account the equality of the powers of the work performed by the external and internal forces, we arrive at the inequality:

$$P < \frac{2h}{\sqrt{3}}\left(\sigma_{s1}^2 k_1^2 + \sigma_{s1}\sigma_{s2}k_1k_2 + \sigma_{s2}^2k_2^2\right)^{1/2} + P_{\text{pl}}, \tag{7}$$

since the ratio of the integrals

$$\left(\frac{h^2}{\sqrt{3}}\int_F\left\{\frac{1}{3}\left[\sigma_{s1}^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \sigma_{s1}\sigma_{s2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \sigma_{s2}^2\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] + \frac{\tau_s^2}{4}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}^{1/2} dF\right) : \int_F w\, dF,$$

taken over the area F occupied by the plan of the shell, determines the load-carrying capacity of the flat plate (P_{pl}). Thus, inequality (7) gives an upper estimate of the load-carrying capacity of the shell. Incidentally, the inequality is also valid in the presence of clamping along the contour, and also in the case of alternating clamping with hinged support; only the value of the load-carrying capacity of the plate P_{pl} will then be different. We shall produce a lower estimate of the shell load-carrying capacity with the aid of a statically possible solution. According to one variant of such a solution, one may set $T_1 = S = T_2 = 0$. Analyzing (1) and (3) in this case, we arrive at the almost obvious conclusion: $P > P_{\text{pl}}$.

Another variant of a statically possible solution is constructed under the assumption that $M_1 = H = M_2 = S = 0$, $T_1 = \sigma_{s1}hC_1$, $T_2 = \sigma_{s2}hC_2$, where

$$C_1 = -\frac{2k_1\sigma_{s1} + k_2\sigma_{s2}}{\sqrt{3}\left(k_1^2\sigma_{s1}^2 + k_1k_2\sigma_{s1}\sigma_{s2} + k_2^2\sigma_{s2}^2\right)^{1/2}}, \qquad C_2 = \frac{C_1}{2} - \left(1 - \frac{3C_1^2}{4}\right)^{1/2}.$$

The corresponding estimate has the form:

$$P > \frac{2h}{\sqrt{3}} \left(\sigma_{s1}^2 k_1^2 + \sigma_{s1}\sigma_{s2} k_1 k_2 + \sigma_{s2}^2 k_2^2\right)^{1/2}.$$

Of course, the estimate of the load-carrying capacity of the shell can also be carried out directly, without resorting to comparison with a plate.

3. An equal-strength (and, at the same time, minimum-weight) shell, under the action of prescribed forces, instantly passes into the yield state ([4],[5]). Consequently, the plasticity coefficient λ does not depend on the coordinates x and y.

The resolving system of equations for such a shell consists of three equilibrium equations (1), the yield condition (3), and three compatibility relations for strains, which have the form:

$$\frac{\partial^2 \varepsilon_1}{\partial y^2} + \frac{\partial^2 \varepsilon_2}{\partial x^2} - \frac{\partial^2 \omega}{\partial x\, \partial y} + k_2 \chi_1 + k_1 \chi_2 = 0, \qquad \frac{\partial \chi_1}{\partial y} = \frac{\partial \chi}{\partial x}, \qquad \frac{\partial \chi_2}{\partial x} = \frac{\partial \chi}{\partial y}.$$

In the latter relations, $\varepsilon_1, \varepsilon_2, \omega, \chi_1, \chi_2, \chi$ must be expressed in terms of forces and moments according to (4), under the assumption that the coefficient λ is constant. The resulting system of 7 equations makes it possible to determine both the unknown forces and moments $(T_1, S, T_2, M_1, H, M_2)$, and the sought shell thickness h, ensuring its uniform strength.

The problem is simplified if one abandons the requirement of instantaneous onset of yielding and regards the stress state of the shell as momentless. The equilibrium equations and the yield condition take the form

$$\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial y} + P_x = 0, \qquad \frac{\partial T_2}{\partial y} + \frac{\partial S}{\partial x} + P_y = 0, \qquad k_1 T_1 + k_2 T_2 - P_z = 0,$$

$$h_2 = \frac{T_1^2}{\sigma_{s1}^2} - \frac{T_1 T_2}{\sigma_{s1}\sigma_{s2}} + \frac{T_2^2}{\sigma_{s2}^2} + \frac{S^2}{\tau_s^2}.$$

In this case, to determine the forces (on the basis of the first three equations) it is natural to use a stress function, and to compute the required shell thickness h from the yield condition.

4. Another approach to the problem of uniform strength may also be proposed: namely, for a given load and shell thickness, to seek a rational outline of its middle surface $z = f(x, y)$. In this case the structure of the corresponding resolving system of equations remains the same. Only some of the equations (equilibrium and compatibility of strains) are modified somewhat, owing to the appearance of additional terms accounting for the

twisting of the sought surface, since it cannot be asserted in advance that the chosen axes x and y will coincide with the lines of principal curvatures. As before, the problem is simplified for momentless shells, whose transition into the plastic state does not necessarily occur instantaneously. In this case the forces are determined on the basis of the first two equilibrium equations and the yield condition by methods of the plane problem of the theory of plasticity (for an orthotropic medium). The third equilibrium equation, in which $k_1 = \partial^2 z/\partial x^2$, $k_2 = \partial^2 z/\partial y^2$, $k_{12} = \partial^2 z/\partial x\,\partial y$, serves to find the sought surface.

For a shell with hinged edges, subjected only to a vertical load ($P_x = P_y = 0$), the force field satisfying the first two equilibrium equations and the yield condition has the form $S = 0$, $T_1 = \sigma_{s1} h$, $T_2 = \sigma_{s2} h$, and the middle surface is determined from the equation:

$$\sigma_{s1} h \frac{\partial^2 z}{\partial x^2} + \sigma_{s2} h \frac{\partial^2 z}{\partial y^2} - P_z = 0,$$

whose solution, for prescribed values of z on the boundary, presents no difficulty. In the case of an isotropic shell ($\sigma_{s1} = \sigma_{s2} = \sigma_s$), the solution obtained, along with uniform strength, also ensures equal stress of the structure. The latter case as an independent problem was considered in ([7,8]).

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Note: Figure translations are in progress. See original paper for figures.