EXTREMAL PROBLEMS ON CLASSES OF ANALYTIC FUNCTIONS HAVING A STRUCTURAL FORMULA

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EXTREMAL PROBLEMS ON CLASSES OF ANALYTIC FUNCTIONS HAVING A STRUCTURAL FORMULA

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The basic internal problems of the theory of analytic functions can be regarded as special cases of the extremal problem formulated below in § 1. In § 2 we give the results of its study for classes of functions with a structural formula. At the same time, in contrast to ([1]), where, in investigating a somewhat less general problem, the method of internal variations was used, we proceed from the structural formulas themselves. The method of variation of a Stieltjes integral that we use in the proof of Theorem 1 goes back to G. M. Goluzin ([2]). Theorem 1 is a generalization of the main theorem of V. A. Zmorovich ([3]) and of the theorems of Yu. E. Alenitsyn ([4]). In § 3 applications are given of the corollaries of Theorem 1 to certain extremal problems posed for classes of starlike, typically real, holomorphic bounded, and other functions.

§ 1. Let B be a finite domain of the complex plane z, and let M be a class, compact in itself with respect to uniform convergence inside B, of functions $f(z)$ analytic in B, admitting in M a homotopy to a fixed function $f_0(z) \in M$. A bounded complex-valued functional $I(f)$ is called **weakly differentiable on the class** M if, for any function $f(z) \in M$ and function $f_*(z) = f(z) + \lambda P(z) + o(\lambda) \in M$ $(\lambda > 0)$, uniformly inside the domain B, differentiable with respect to λ at $\lambda = 0$, there exists the functional derivative

$$\lim_{\lambda \to +0} \frac{I(f_n) - I(f)}{\lambda} = \Phi[f, P],$$

representable in the form of the following sum of two linear functionals $\Phi^{(1)}[f, P]$ and $\Phi^{(2)}[f, P]$ on the class of meromorphic functions P:

$$\Phi[f, P] = \Phi^{(1)}[f, P] + \Phi^{(2)}[f, P].$$

A continuous system $I(f) = \{I_n(f)\}$ $(n = 1, 2, ...)$ of weakly differentiable functionals with a convergent sum of squares of moduli is called **weakly dif-**

ferentiable if the sum of squares of the moduli of the functional derivatives $\Phi_n[f,P]$ for $I_n(f)$ on the class M converges. We give three examples of weakly differentiable functionals. In the absence of additional restrictions it is assumed that J_n are given analytic functions.

1. Let $z_1, z_2, \dots, z_p$ $(p = 1, 2, \dots)$ be fixed points in B, and let $u_{mj} = f^{(m)}(z_j)$, $v_{mj} = \overline{u_{mj}}$ $(j = 1, 2, \dots, p;\ m = 0, 1, \dots, s_j)$. Consider on the class M the functional

$$I_n(f) = J_n(u_{01}, v_{01}, \dots, u_{s_1 1}, v_{s_1 1}; \dots; u_{0p}, v_{0p}, \dots, u_{s_p p}, v_{s_p p}). \tag{1}$$

For it

$$\Phi_n[f,P] = \sum_{j=1}^{p} \sum_{m=0}^{s_j} \left(a_{mj}^{(n)} \frac{d^m P(z_j)}{dz^m} + b_{mj}^{(n)} \overline{\frac{d^m P(z_j)}{dz^m}} \right),$$

where

$$a_{mj}^{(n)} = \partial J_n / \partial u_{mj}, \qquad b_{mj}^{(n)} = \partial J_n / \partial v_{mj} \tag{2}$$

are quantities depending only on the choice of J_n and on f.

2. Let $z_1, z_2, \dots, z_p$ $(p = 1, 2, \dots)$ be fixed points in the z-plane, for each of which there exists an annulus $K_j:\ r_j < |z - z_j| < R_j$ $(r_j \geq 0;\ j = 1, 2, \dots, p)$, lying entirely in B. We shall denote the coefficients of the expansion of an arbitrary function $\varphi(z)$, holomorphic in K_j, into a Laurent series in this annulus by $\{\varphi\}_{mj}$:

$$\varphi(z) = \sum_{m=-\infty}^{\infty} \{\varphi\}_{mj} (z - z_j)^m.$$

Consider on the class M the functional

$$I_n(f) = J_n(\{f\}_{s_1 1}, \{\overline{f}\}_{s_1 1}, \dots, \{f\}_{q_1 1}, \{\overline{f}\}_{q_1 1}; \dots; \{f\}_{s_p p}, \{\overline{f}\}_{s_p p}, \dots, \{f\}_{q_p p}, \{\overline{f}\}_{q_p p}) \tag{3}$$

$$(s_j < q_j;\ j = 1, 2, \dots, p),$$

which depends analytically on the coefficients of the Laurent expansions of $f(z)$ in the annuli K_j. The functional derivative for $I_n(f)$ has the form

$$\Phi_n[f, P] = \sum_{j=1}^{p} \sum_{m=s_j}^{q_j} \left[a_{mj}^{(n)} \{P\}_{mj} + b_{mj}^{(n)} \{\overline{P}\}_{mj} \right],$$

where

$$a_{mj}^{(n)} = \partial J_n / \partial \{f\}_{mj}, \qquad b_{mj}^{(n)} = \partial J_n / \partial \{\overline{f}\}_{mj} \tag{4}$$

are quantities depending only on the choice of J_n and on f.

3. Let σ be a domain or a rectifiable curve in B. Just as (1) and (3), the functional

$$I_n(f) = \int_\sigma J_n(f(z), \overline{f(z)}, \dots, f^{(s)}(z), \overline{f^{(s)}(z)})\, d\sigma_z \tag{5}$$

is weakly differentiable on M.

The set D of values $I(f) = \{I_n(f)\}$ in the space l^2, as f ranges over the class M, is connected and closed. A point $I^0 = \{I_n^0\}$ of the boundary ∂D of the domain D of values $I(f)$ is called nonsingular[(5)] if there exists a point $I^e = \{I_n^e\}$ such that

$$\rho(I^0, I^e) = \min_{I \in D} \rho(I, I^e),$$

where $\rho(x, y)$ is the distance between x and y in l^2 $(x, y \in l^2)$. The remaining points of ∂D, called singular, turn out to be limiting points for nonsingular boundary points. Functions from M that introduce into D nonsingular boundary points are called boundary functions (with respect to $I(f)$ on the class M). The problem arises: to find the set D of values $I(f)$ on M and to single out in M the functions that are boundary functions with respect to $I(f)$.

§ 2. Denote by M the class of functions

$$f(z) = \sum_{k=1}^{l} \int_a^b g_k(z, t)\, d\mu_k(t), \tag{6}$$

given in the form of a sum of Stieltjes integrals. In (6), $g_k(z, t)$ are functions regular in B for each t in the interval $[a, b]$, uniformly differentiable with respect to t inside B, and, moreover, $\psi_k(z, t) = g'_{kt}(z, t)$ is absolutely integrable on $[a, b]$ for fixed $z \in B$; $\mu_k(t)$ are arbitrary real nondecreasing functions on $[a, b]$ with total variation equal to one.

Theorem 1. Let $I(f) = \{I_n(f)\}$ be a weakly differentiable system of functionals on the class M, for which

$$\Phi_n\left[f, \int \psi_k(z,t)\gamma_k(t)\,dt\right] = \int \Phi_n[f, \psi_k(z,t)\gamma_k(t)]\,dt,$$

where $\gamma_k(t)$ is a function of bounded variation on $[a,b]$. Suppose that the sum of scalar products

$$(\Phi^{(1)}[f, \psi_k(z,t)], N) + (\Phi^{(2)}[f, \psi_k(z,t)], \overline{N})$$

of the elements $\Phi^{(1)}$ and $N \equiv I^0 - I^e$, $\Phi^{(2)}$ and $\overline{N}$, converges uniformly on $[a,b]$ to a continuous function $\varphi_k(t)$. Then all boundary functions with respect to $I(f)$ are contained in the family of functions

$$f(z) = \sum_{k=1}^{l} \sum_{s=1}^{L_k} \mu_k^{(s)} g_k(z, t_k^{(s)}).$$

Here $t_k^{(s)} \in [a,b]$, $\mu_k^{(s)} \geq 0$,

$$\sum_{s=1}^{L_k} \mu_k^{(s)} = 1,$$

with $1 \leq L_k \leq [(M_k+1)/2]$, where M_k is the number of solutions of the equation $\operatorname{Re}\varphi_k(t) = 0$ on $[a,b]$.

Let us outline the proof of the theorem. If $f(z)$ is an extremal function with respect to $I(f)$ on M, then $\operatorname{Re}(\Phi[f,P], N) \geq 0$. Generalizing the arguments of G. M. Goluzin ([2]), one can show that, for $0 \leq \theta_k \leq 1$, $0 \leq \lambda_k \leq 1$, the class M contains the functions

$$f_*(z) = f(z) + \sum_{k=1}^{l} \lambda_k \int_{\alpha_k}^{\beta_k} \psi_k(z,t) v_k(t)\,dt, \tag{7}$$

where $[\alpha_k, \beta_k] \subset [a,b]$,

$$v_k(t) = \mu_k(t) - \{\mu_k(\alpha_k - 0) + \theta_k[\mu_k(\beta_k + 0) - \mu_k(\alpha_k - 0)]\}$$

on (α_k, β_k), and $v_k(t) = 0$ on $[a, \alpha_k] \cup [\beta_k, b]$. Moreover, if α_k and β_k are two arbitrary points of discontinuity of the function $\mu_k(t)$ on $[a,b]$, then, for numbers λ_k sufficiently small in absolute value, the class M contains the function

$$f_*(z) = f(z) + \sum_{k=1}^{l} \lambda_k [g_k(z, \alpha_k) - g_k(z, \beta_k)]. \tag{8}$$

The inequality indicated above, together with the variation (7), makes it possible to derive a certain equality that analytically characterizes $f(z)$. It is established that this equality is fulfilled only in the case when the functions $\mu_k(t)$ corresponding to $f(z)$ are step functions with a number of jumps not exceeding the number of zeros of $\operatorname{Re}\varphi_k(t)$ on $[a, b]$. Then, using (8), this estimate can be improved by a factor of two.

Corollary 1. The boundary ∂D of the domain D of values of the functional $I(f)$, satisfying the conditions of Theorem 1, belongs to the set

$$I(f) = I\left(\sum_{k=1}^{l}\sum_{s=1}^{L_k}\mu_k^{(s)} g_k(z, t_k^{(s)})\right).$$

If the functionals $I_n(f)$ are linear, then D is a convex set.

Corollary 2. The extremal functions, with respect to a finite system of functionals prescribed on the class M,

$$I(f) = \{I_n(f)\} \quad (n = 1, 2, \dots, r),$$

with $I_n(f)$ defined in accordance with (1) [or with (3)], have the form

$$f(z) = \sum_{k=1}^{l}\sum_{s=1}^{L_k}\mu_k^{(s)} g_k(z, t_k^{(s)}).$$

Here $t_k^{(s)} \in [a, b]$, $\mu_k^{(s)} \geq 0$, and

$$\sum_{s=1}^{L_k}\mu_k^{(s)} = 1,$$

with $1 \leq L_k \leq [(M_k + 1)/2]$, and M_k is the number of zeros on $[a, b]$ of the corresponding functions

$$\varphi_k(t) = \operatorname{Re}\sum_{n=1}^{r}\sum_{j=1}^{p}\sum_{m=0}^{s_j}\gamma_{mj}^{(n)}\frac{d^m}{dz^m}\psi_k(z_j, t),$$

$$\varphi_k(t) = \operatorname{Re}\sum_{n=1}^{r}\sum_{j=1}^{p}\sum_{m=s_j}^{q_j}\gamma_{mj}^{(n)}\{\psi_k\}_{mj},$$

where

$$\gamma_{mj}^{(n)} = a_{mj}^{(n)} \overline{N}_n + b_{mj}^{(n)} N_n, \qquad N_n = I_n^0 - I_n^e,$$

and $a_{mj}^{(n)}, b_{mj}^{(n)}$ are computed by formulas (2) [respectively, by formulas (4)].

Thus, the family containing all extremal functions depends on no more than

$$M_1 + \dots + M_l - l$$

real parameters.

§ 3. By the order of the functional (1) we shall mean the number $Q = s_1 + \dots + s_p$. In Theorems 2-5, $I(f)$ is a finite system of functionals $I_n(f)$ $(n = 1, 2, \dots, r)$ of the form (1), and, consequently, $D \subset E^r$.

Theorem 2. Let the system $I(f)$ be given on the class S^* of functions $w = f(z)$, $f(0) = 0$, $f'(0) = 1$, mapping the disk $|z| < 1$ onto domains star-shaped with respect to the point $w = 0$. Then all boundary functions for $I(f)$ are contained in the family

$$f(z) = z \prod_{k=1}^{N} \left(1 - e^{-i\alpha_k} z\right)^{-\tau_k},$$

depending on the real parameters τ_k and α_k $(k = 1, 2, \dots, N)$, subject to the conditions: $\tau_k \geq 0$, $\tau_1 + \dots + \tau_N = 2$, $\alpha_1 < \alpha_2 < \dots < \alpha_N < \alpha_1 + 2\pi$, with $N \leq Q + p$.

As shown in (6), for $p = r = 1$ the estimate for N cannot be improved.

Theorem 3. Suppose that the system $I(f)$ is given on the class T_r of functions typically real in the disk $|z| < 1$,

$$f(z) = \frac{1}{\pi} \int_0^{\pi} \frac{z}{1 - 2z\cos t + z^2} \, d\mu(t).$$

Then all boundary functions for $I(f)$ are contained in the family

$$f(z) = \sum_{k=1}^{N} \lambda_k \frac{z}{1 - 2z\tau_k + z^2},$$

depending on the real parameters λ_k and τ_k $(k = 1, 2, \dots, N)$, subject to the conditions $\lambda_k \geq 0$, $\lambda_1 + \dots + \lambda_N = 1$, $-1 \leq \tau_1 < \dots < \tau_N \leq 1$, with $N \leq Q + 2p$.

Theorem 4. Let the system $I(f)$ be defined on the class K of functions regular in the disk $|z| < 1$, $w = f(z)$, $f(0) = 1$, with positive real part. Then the boundary functions for $I(f)$ are contained in the family

$$f(z)=\sum_{k=1}^{N}\lambda_k\frac{e^{i\alpha_k}+z}{e^{i\alpha_k}-z},$$

depending on the real parameters λ_k and α_k ($k=1,2,\dots,N$), subject to the conditions $\lambda_k\geq 0$, $\lambda_1+\dots+\lambda_N=1$, $\alpha_1<\alpha_2<\dots<\alpha_N<\alpha_1+2\pi$, with $N\leq Q+2p-1$.

Theorem 5. Let the system $I(f)$ be defined on the class of functions regular in the disk $|z|<1$, $w=f(z)$, with majorant $F(z)$*.

Then the nonzero boundary functions with respect to $I_t(f)$ have the form $f(z)=F(b_N(z))$, where $b_N(z)$ is the Blaschke product

$$b_N(z)=e^{\gamma}\prod_{k=1}^{N}\frac{z-\alpha_k}{1-\overline{\alpha}_k z},\qquad \gamma=\text{const},\qquad \operatorname{Re}\gamma\leq 0,$$

with a finite number $N\leq Q+2p-1$ of zeros α_k, $0=\alpha_1\leq\dots\leq\alpha_N<1$.

Analogous theorems hold for the corresponding classes of functions in a circular annulus, and also for finite systems composed only of functionals of the form (1), or mixed systems including both functionals of the form (1) and functionals of the form (3).

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CITED LITERATURE

[1] I. A. Aleksandrov, DAN, **151**, No. 5, 999 (1963).
[2] G. M. Goluzin, Uchen. zap. LGU, No. 144, issue 23 (1952).
[3] V. A. Zmorovich, Ukr. matem. zhurn., **4**, No. 3, 176 (1952).
[4] Yu. E. Alenitsyn, Vestn. LGU, **7**, issue 2, 25 (1962).
[5] N. A. Lebedev, Vestn. LGU, **8**, 29 (1955).
[6] I. A. Aleksandrov, V. V. Chernikov, Sibirsk. matem. zhurn., **4**, No. 6, 1201 (1963).

* In other words, for $f(z)$ there exists a function $\omega(z)$, holomorphic in $|z|<1$, $\omega(z)\not\equiv 0$, $\omega(0)=0$, $|\omega|<1$, such that $f(z)=F(\omega(z))$.

Note: Figure translations are in progress. See original paper for figures.