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A GENERALIZATION OF THE DU BOIS-REYMOND THEOREM TO BOUNDED ORTHONORMAL SYSTEMS OF FUNCTIONS

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§ 1. In a recent note ([1]) we established the existence of an orthonormal system $\{\theta_n(x)\}$, consisting of uniformly bounded functions and possessing the property that every continuous function (moreover, every function in L^2) is represented by a Fourier series converging to it almost everywhere.

The present note contains a result of the opposite character. It turns out that it is impossible to construct an orthonormal system of uniformly bounded functions with respect to which every continuous function would be represented by a uniformly convergent (or even merely everywhere convergent) Fourier series.

The starting point in this circle of questions is the classical result of Du Bois-Reymond, according to which the Fourier series of a continuous function with respect to the trigonometric system may diverge at some point. This result prompted Haar to raise the question of the existence of an orthonormal system giving, for every continuous function, an everywhere convergent Fourier expansion. Haar gave a positive answer to this question. He constructed a complete orthonormal system forming a basis in the space C of continuous functions. The latter means that every continuous function is expanded in a uniformly convergent series with respect to the Haar system. Later Franklin constructed an orthonormal basis in C, consisting of continuous functions (whereas the Haar functions are step functions). Another classical complete orthonormal system—the Walsh system—like the trigonometric system, does not form a basis in C.

It should be noted that the trigonometric system and the Walsh system consist of functions bounded in the aggregate; the Haar system, however, does not possess this property. In this connection the following problem arose: does there exist an orthonormal system bounded in the aggregate with respect to which every continuous function is expanded in a Fourier series converging to it everywhere (uniformly)?

This question was formulated by P. L. Ul' yanov in survey articles ([2], [3]) among the unsolved problems of the theory of orthogonal series. The principal result of the present note is the following theorem, which gives an answer to this question.

Theorem 1. Let $\{\varphi_n(x)\}$ be an orthonormal system in $L^2[0,1]$, consisting of uniformly bounded functions. Then there exists a continuous function $f(x)$ whose Fourier series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x), \qquad a_n = \int_0^1 f(x)\varphi_n(x)\,dx \tag{1}$$

diverges at some point.

This result is a direct generalization of the Du Bois-Reymond theorem.

It follows immediately from Theorem 1:

Theorem 2. *There is no orthonormal system, bounded in the aggregate, that forms a basis in the space C.*

From the results given below there also follows the following theorem.

Theorem 3. *There is no orthonormal system, bounded in the aggregate, that forms a basis in the space L.*

It is useful to note that in the spaces L^p $(1 < p < \infty)$ there exists a basis consisting of orthonormal functions bounded in the aggregate, namely the trigonometric system (M. Riesz).

§ 2. The results of this note follow from the following theorem.

Theorem 4. *Let $\{\varphi_n(x)\}$ be an orthonormal system consisting of uniformly bounded functions. Then, if the sequence $\{c_n\}$ satisfies the condition*

$$\overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} c_k^2 > 0, \tag{2}$$

then the series

$$\sum_{k=1}^{\infty} c_k \varphi_k(x) \tag{3}$$

is unbounded in the metric L, i.e.

$$\overline{\lim}_{n\to\infty} \int_0^1 \left| \sum_{k=1}^{n} c_k \varphi_k(x) \right| dx = \infty.$$

Without touching on the proof of this theorem, we shall indicate here some consequences.

First of all, Theorem 4 gives information about the behavior of the Lebesgue functions of an arbitrary orthonormal system bounded in the aggregate. Recall that the Lebesgue functions of the system $\{\varphi_n(x)\}$ are defined as follows:

$$L_n(x) = \int_0^1 \left|\sum_{k=1}^{n} \varphi_k(x)\varphi_k(t)\right| dt.$$

The following assertion is easily derived from Theorem 4.

Theorem 5. *Let the orthonormal system $\{\varphi_n(x)\}$ consist of uniformly bounded functions. Then the Lebesgue functions of this system satisfy the condition*

$$\overline{\lim}_{n\to\infty} L_n(x) = \infty, \qquad x \in E, \quad \mu E > 0. \tag{4}$$

The fact that the upper limit here cannot be replaced by the lower one is evident from the example of the Walsh system. It should be further noted that even for complete systems condition (4) is not necessarily satisfied almost everywhere. Namely, the following theorem is true:

Theorem 6. *There exists a complete system $\{\varphi_n(x)\}$, bounded in the aggregate, for which the Lebesgue functions $L_n(x)$ satisfy the condition*

$$L_n(x) < C, \qquad x \in U, \quad \mu U > 0.$$

These results answer the question posed in ([2]) (p. 140). The proof of the last theorem is based on a construction close to the one used by us in ([4]).

It is obvious that Theorem 5 immediately implies Theorems 1-3, and in somewhat stronger formulations. Namely: the set of functions expandable in a uniformly convergent Fourier series with respect to a fixed orthonormal system bounded in the aggregate has first

a category in the space C; moreover, there exists a set E, $\mu E > 0$, such that for each fixed point $x \in E$ the set of functions whose Fourier series (1) is bounded at the point x is of first category in C.

Finally, one can prove the following assertion, strengthening Theorem 1.

Theorem 7. *For any orthonormal system $\{\varphi_n(x)\}$ consisting of uniformly bounded functions, one can indicate a continuous function whose Fourier series (1) diverges on a set of cardinality continuum.*

It is useful to compare the results just formulated with the following assertion, close to Theorem 6.

Theorem 6′. *Let $0 < \gamma < 1$. Then there exists a complete system $\{\varphi_n(x)\}$, bounded in the aggregate, possessing the property that for every continuous function $f(x)$ the Fourier series (1) converges uniformly on the interval $[0, \gamma]$.*

§ 3. In connection with Theorems 2 and 3 it is natural to pose the question: how must an arbitrary orthonormal system forming a basis in the space C (or L) be constructed? To what extent must it resemble the "standard" basis—the Haar system? We present some results in this direction. First of all, the following assertion is valid, strengthening Theorem 2.

Theorem 8. *Let $\{\varphi_n(x)\}$ be an orthonormal system forming a basis in C. Then, for all $p > 2$, $1 \leq q < 2$, the relations*

$$\lim_{n\to\infty} \|\varphi_n(x)\|_{L^p} = \infty, \qquad \lim_{n\to\infty} \|\varphi_n(x)\|_{L^q} = 0$$

hold

(the case $p = \infty$ gives Theorem 2).

This property is characteristic of the Haar system. However, orthonormal bases in C may also differ substantially from the Haar system. The first assertion of this kind, as an intermediate result, was already contained in note (1). Namely:

Theorem 9. *There exists an orthonormal system $\{\varphi_n(x)\}$, forming a basis in C and containing a subsystem $\{\varphi_{n_k}(x)\}$ of uniformly bounded functions.*

Finally, the following proposition can be proved.

Theorem 10. *There exists an orthonormal system $\{\varphi_n(x)\}$, forming a basis in C and satisfying the condition*

$$|\varphi_n(x)| < f(x) \qquad (n = 1, 2, \dots;\ x \in [0, 1]),$$

where $f(x)$ is a finite function everywhere.

Theorems 9 and 10 limit the possibilities of generalizing Theorem 2.

§ 4. Theorem 4 makes it possible to derive some consequences relating to another circle of questions, namely, to series with monotone coefficients.

It is known that a trigonometric series with coefficients decreasing monotonically to zero need not be a Fourier-Lebesgue series (Sidon). This result was transferred in (5) to series with respect to the Walsh system. We showed in (4) that for arbitrary complete systems such a result does not hold. It turns out, however, that such a theorem is valid for any complete system bounded in the aggregate. First of all, let us note the following immediate consequence of Theorem 4.

Corollary. *If an orthonormal system $\{\varphi_n(x)\}$ consists of uniformly bounded functions, then the condition*

$$\overline{\lim_{n\to\infty}} \int_0^1 \left| \sum_{k=1}^{n} \varphi_k(x) \right| dx = \infty$$

is fulfilled.

Hence the following assertion follows easily.

Theorem 11. Let the orthonormal system $\{\varphi_n(x)\}$ consist of uniformly bounded functions. Then there exists a series (3) with coefficients c_n, monotonically tending to zero, which diverges in the metric L.

In turn, from this theorem, by virtue of known arguments connected with the application of Banach's closed graph theorem (see, for example, [6], p. 271, Orlicz' s lemma), the following theorem follows.

Theorem 12. Let $\{\varphi_n(x)\}$ be an orthonormal system consisting of uniformly bounded functions and complete in the space L. Then there exists a series (3) with coefficients monotonically tending to zero which is not a Fourier–Lebesgue series.

We note that the completeness condition imposed in the last result is essential, as is seen from the example of lacunary systems; indeed, for, say, a lacunary trigonometric system, as is known, any series with coefficients tending to zero is a Fourier–Lebesgue series (Banach).

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