ON THE REPRESENTATION OF ARBITRARY ENTIRE FUNCTIONS BY DIRICHLET SERIES

MATHEMATICS

1965

SovietRxiv

Abstract

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UDC 517.522.6

MATHEMATICS

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ON THE REPRESENTATION OF ARBITRARY ENTIRE FUNCTIONS BY DIRICHLET SERIES

(Presented by Academician Yu. V. Linnik on 19 IV 1965)

In the note [1] the question was the representation of arbitrary analytic functions by Dirichlet series with complex exponents in a finite convex domain. Here results will be given concerning the representation of arbitrary entire functions from a certain class by Dirichlet series in the whole plane.

Let

$$L(\lambda) = \sum_{0}^{\infty} c_n \lambda^n$$

be an entire function of finite order $\rho > 1$, satisfying the following condition: there exists a system of circles $|\lambda| = r_k$, $r_k \uparrow \infty$, such that $\ln|L(re^{i\varphi})| > r^{\rho-\varepsilon}$, $r = r_k$, $k > K(\varepsilon)$, $\varepsilon > 0$ arbitrary. Let $\lambda_1, \lambda_2, \dots$ be the distinct zeros of $L(\lambda)$, arranged in order of nondecreasing moduli, and let $p_1, p_2, \dots$ be their respective multiplicities. Take an arbitrary entire function $f(z)$ of order ν, satisfying the condition $\nu < \rho/(\rho-1)$, and introduce the function

$$\omega_f(\mu) = \omega(\mu) = \sum_{n=1}^{\infty} c_n \left[f^{(n-1)}(0) + \mu f^{(n-2)}(0) + \dots + \mu^{n-1} f(0) \right].$$

It is easy to see that $\omega(\mu)$ is an entire function. Put

$$P_k(z) e^{\lambda_k z} = \frac{1}{2\pi i} \int_{C_k} \frac{\omega(\mu) e^{\mu z}\, d\mu}{L(\mu)} \qquad (k = 1, 2, \dots),$$

where C_k is a closed contour inside which λ_k lies and there are no other zeros of $L(\lambda)$.

Theorem 1. *For an entire function $f(z)$ of order $\nu < \rho/(\rho-1)$ the representation*

$$f(z) = \sum_{m=1}^{\infty} f_m(z), \qquad f_1(z) = \sum_{|\lambda_k|<r_1} P_k(z) e^{\lambda_k z}, \qquad f_m(z) = \sum_{r_{m-1}<|\lambda_k|<r_m} P_k(z) e^{\lambda_k z}$$

$$(m \geq 2), \tag{1}$$

takes place, and the series converges absolutely and uniformly inside the plane. Moreover, for any z *the estimate is valid*

$$\left| f(z) - \sum_{m=1}^{n} f_m(z) \right| < A(\varepsilon) e^{-r_n^{\rho-\varepsilon}} \exp |z|^{p+\varepsilon}, \qquad p = \frac{\rho}{\rho - 1} \quad (n = 1, 2, \ldots), \tag{2}$$

where $\varepsilon > 0$ *is arbitrary.*

Theorem 2. *Suppose that the function* $L(\lambda)$ *additionally satisfies the following conditions: 1) all zeros* $\lambda_1, \lambda_2, \ldots$ *are simple and their number in the ring* $r_{k-1} < |\lambda| < r_k$ *does not exceed a certain fixed number, one and the same for all* $k = 1, 2, \ldots$*; 2) there exists a constant* q *such that* $r_k / r_{k-1} < q$ $(k = 2, 3, \ldots)$*; 3) there exist constants* A *and* h*,* $h < \rho$*, such that for any* λ_m *and* λ_n*,* $m \neq n$*, from the ring* $r_{k-1} < |\lambda| < r_k$

we have $|\lambda_m - \lambda_n| > A \exp(-r_k^h)$. Then the representation

$$f(z) = \sum_{m=1}^{\infty} a_m e^{\lambda_m z}, \qquad a_m = \frac{\omega(\lambda_m)}{L'(\lambda_m)}. \tag{3}$$

As an example of a function $L(\lambda)$, when the representation (3) holds, one may take the function of order ρ

$$L(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^m}{\mu_n^m}\right), \qquad \mu_n = n^{1/\rho} \quad (n = 1, 2, \ldots),$$

where m is an integer $> 2\rho$. Let us note that in estimate (2) the quantity ρ in general cannot be replaced by a smaller quantity. Indeed, it is shown in (^{2}) that if

$$\left| f(z) - \sum_{m=1}^{n} f_m(z) \right| < A \exp |z|^h, \qquad h < \frac{\rho}{\rho - 1},$$

then the function $f(z)$ satisfies the equation

$$M(f) \equiv \sum_{0}^{\infty} c_m f^{(m)}(z) = 0;$$

but not every function satisfies this equation; for example, if $\lambda \neq \lambda_n$ $(n = 1, 2, ...)$, then

$$M(e^{\lambda z}) = e^{\lambda z} L(\lambda) \neq 0.$$

Theorem 3. If $f(z)$ is an entire function of order $\nu < \rho/(\rho-1)$ and $M(f) = 0$, then instead of (2) we have the better estimate

$$\left| f(z) - \sum_{m=1}^{n} f_m(z) \right| < A(\varepsilon) e^{-r_n^{\rho-\varepsilon}} \exp |z|^{\nu+\varepsilon} \qquad (n = 1, 2, ...).$$

This estimate is of the same type as in the paper (³), but more precise. The difference between functions $f(z)$ that do not satisfy the equation $M(y) = 0$ and functions $f(z)$ that do satisfy such an equation is manifested not only in the different rates of convergence of the series (1). As will be seen from what follows, it is also manifested in many other facts.

Theorem 4. Suppose the conditions of Theorem 2 are satisfied, and let $f(z)$ be an entire function of order $\nu < \rho/(\rho-1)$ with expansion (3). Then the quantity

$$\overline{\lim_{n\to\infty}} \frac{\ln |\lambda_n|}{\ln \ln |1/a_n|}, \tag{4}$$

which characterizes the rate at which a_n tends to zero, is equal to $1/\rho$ if $M(f) \not\equiv 0$, and is equal to $(\nu-1)/\nu$ $((\nu-1)/\nu < 1/\rho)$ if $M(f) \equiv 0$.

To this result one may add that if the quantity (4) is equal to $(\nu-1)/\nu$, where $\nu < \rho/(\rho-1)$, then (under the conditions of Theorem 2) the function $f(z)$ defined by the series (3) has order ν and satisfies the equation $M(f) = 0$.

Theorem 5. Suppose the conditions of Theorem 2 are satisfied, and let $f(z)$ be an entire function of order $\nu < \rho/(\rho-1)$ with expansion (3). If $M(f) = 0$, then the function

$$F(z) = \sum_{m=1}^{\infty} b_m e^{\lambda_m z}, \qquad |b_m| = |a_m| \quad (m = 1, 2, ...) \tag{5}$$

has order ν; if $M(f) \not\equiv 0$, then there exists a function $F(z)$ of the form (5) whose order is equal to $\rho/(\rho-1) > \nu$.

Consider the maximum term $\mu(r)$ of the series (3). By definition,

$$\mu(r) = \max_{n\geqslant 1}\max_{|z|=r}\left|a_n e^{\lambda_n z}\right| = \max_{n\geqslant 1}|a_n|e^{|\lambda_n|r}.$$

Theorem 6. *Under the conditions of Theorem 2, the quantity*

$$\varlimsup_{r\to\infty}\frac{\ln\ln\mu(r)}{\ln r},$$

which characterizes the growth of the maximum term, is equal to the order ν *of the function* $f(z)$ *if* $M(f)\equiv 0$*, and is equal to* $\rho/(\rho-1)$ $(\rho/(\rho-1)>\nu)$ *if* $M(f)\not\equiv 0$*.*

The following assertion concerns the question of the existence of a nontrivial expansion of zero.

Theorem 7. *Let* $\lambda_1, \lambda_2, \ldots$ *be the zeros of a function* $L(\lambda)$ *satisfying the conditions of Theorem 2. Then, in the class of series* $\sum_0^\infty a_n e^{\lambda_n z}$ *for which the quantity (4) is less than* $1/\rho$*, there is no series whose sum would be zero (it is assumed that not all coefficients are equal to zero); on the other hand, in the class of series for which the quantity (4) is equal to* $1/\rho$*, there exist series whose sums are equal to zero.*

Let us indicate the idea of the proof of Theorem 1, the basic one in this note. Put

$$\Phi_k(z,\lambda) = L(\lambda)\frac{1}{2\pi i}\int_{|\mu|=r_k}\frac{e^{z\mu}\,d\mu}{(\mu-\lambda)L(\mu)}$$

for λ lying inside the circle $|\mu| = r_k$. For other λ we define this function by analytic continuation. It is not difficult to verify that $\Phi_k(z,\lambda)$, as a function of the variable λ, is an entire function of order not greater than ρ. Expand it in the series

$$\Phi_k(z,\lambda) = \sum_{m=0}^{\infty} A_m^{(k)}(z)\lambda^m.$$

It turns out that for any function $f(z)$ of order $\nu < \rho/(\rho-1)$ the relation

$$f(z) - \frac{1}{2\pi i}\int_{|\mu|=r_k}\frac{\omega_f(\mu)e^{\mu z}\,d\mu}{L(\mu)} = \sum_{m=0}^{\infty} A_m^{(k)}(z)f^{(m)}(0) \tag{6}$$

holds.

For the coefficients $A_m^{(k)}(z)$ an estimate is found; from it, in particular, it follows that the right-hand side in (6) tends to zero as $k\to\infty$.

Finally, consider the question of the minimal number of rays on which the exponents λ_n must be placed in order that, for example, the function $f(z) \equiv 1$ could be represented in the whole plane by the Dirichlet series

$$\sum_{1}^{\infty} a_n e^{\lambda_n z}.$$

Under the condition $\lim_{n\to\infty} \ln n/\lambda_n = 0$, which ensures absolute convergence of the series, it is necessary for this, as shown in (^{1}), that the number of rays m be $\geqslant 3$. Let $m = 3$. It was noted above (see the example after Theorem 2) that the function

$$L(\lambda) = \prod_{1}^{\infty} (1 - \lambda^3/\mu_n^3), \qquad \mu_n = n^{1/\rho}$$

(its zeros are arranged on three rays), for $\rho < 3/2$, satisfies the conditions of Theorem 2 and, consequently, $f(z) \equiv 1$ can be represented by the series (3).

The coefficients of this series, according to Theorem 4, are such that

$$\varlimsup_{n\to\infty} \frac{\ln|\lambda_n|}{\ln\ln\left|\frac{1}{a_n}\right|} \leq \frac{1}{\rho}. \tag{7}$$

It turns out that if the Λ_n lie on the three rays $\arg z = 0$, $\arg z = \pm 2\pi/3$, and $\lim_{n\to\infty} \ln n/\lambda_n = 0$, then $f(z) \equiv 1$ cannot be represented by a Dirichlet series with coefficients a_n satisfying condition (7) with $\rho > 3/2$. Indeed, suppose that such a representation exists. We split the Dirichlet series into three series: in the first series we collect the terms with $\arg \lambda_n = 0$, in the second we collect the terms with $\arg \lambda_n = 2\pi/3$, and in the third those with $\arg \lambda_n = -2\pi/3$. Let the sums of these series be $f_1(z)$, $f_2(z)$, $f_3(z)$. These functions, by virtue of (7), have orders $\leq \rho/(\rho - 1)$. On the basis of the facts that $f_1 + f_2 + f_3 = 1$ and that the functions $f_j(z)$ are bounded in the corresponding half-planes, it is not difficult to observe that the $f_j(z)$ are bounded on the six rays $\arg z = \varphi_k$, $\varphi_1 = \pi/6$, the angles between which are equal to $\pi/3$. Hence, since $\rho/(\rho-1) < 3$ for $\rho > 3/2$, it follows by the Phragmén-Lindelöf theorem that $f_j(z) = \text{const}$. But then $f_j(z) \equiv 0$, and the equality $f_1 + f_2 + f_3 = 1$ is impossible. Thus, if one wishes to preserve condition (7) with $\rho > 3/2$, it is necessary to take $m > 3$ rays. The example given after Theorem 2 shows that it is sufficient to take $m > 2\rho$ rays.

Received
8 IV 1965

References

[1] A. F. Leont’ ev, *DAN*, **164**, No. 1 (1965).
[2] I. I. Repin, *Matem. sbornik*, **36** (78), 1, 3 (1955).
[3] Yu. N. Frolov, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **64**, 294 (1961).

Note: Figure translations are in progress. See original paper for figures.