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Abstract

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A Theorem on the Density of Finite Functions in Weighted Classes

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Theorems on the density of the set of finite functions in spaces without weight were first proved by S. L. Sobolev in (^{1}, ^{2}). In (^{4}) an analogous result was established for spaces with weight, when $\Omega = E_n$. In the present article more general results are obtained both for bounded and for unbounded domains having a simple boundary in the sense of S. L. Sobolev. Applications are given to the solution of one nonlinear system of differential equations.

1. Let Ω be a domain in the space E_n of points $x = (x_1, x_2, \dots, x_n)$, and suppose that the following conditions are fulfilled:
2. $\Omega = \Omega_1 \times \dots \times \Omega_{P_0}$ $(1 \le P_0 \le 2)$, where Ω_k $(k = 1, 2, \dots, P_0)$ is a domain in the space E_{q_k} of the variables $\tilde{x}_k = (x_{n_{k-1}+1}, \dots, x_{n_k})$ $(0 = n_0 < n_1 < \dots < n_{P_0} = n,\ q_k = n_k - n_{k-1})$, whose boundary S_k is bounded in E_{q_k} and can be divided into a finite number of smooth manifolds $S_k^{(1)}, \dots, S_k^{(A_k)}$ of dimensions $q_k - s_k^{(1)}, \dots, q_k - s_k^{(A_k)}$, respectively. Here and below we introduce the notation:

$$\overline{x}_k = (x_1, \dots, x_{n_{k-1}}, x_{n_k+1}, \dots, x_n).$$

3. Let $T_{\rho_{kj}}$ be the set of those points in Ω_k which lie on the normals to $S_k^{(j)}$ $(j = 1, \dots, A_k)$ planes of dimension $s_k^{(j)}$ at distances $< \rho_{kj}$. Introduce in the space of the variables $y_1, y_2, \dots, y_{q_k}$ cylindrical coordinates with center on the axis

$$y_1 = y_2 = \dots = y_{s_k^{(j)}} = 0.$$

By $\widetilde{\varphi}_{kj}$ and $\tilde{y}_{kj}$ we denote, respectively, the systems of angular coordinates and coordinates of the axis thereby obtained; put

$$\rho_{kj}^2 = y_1^2 + \dots + y_{s_k^{(j)}}^2.$$

The set $T_{\rho_{kj}}$, starting with a sufficiently small $\rho_{kj}^{(0)} > 0$, is mapped by means of a smooth nondegenerate transformation with derivatives bounded up to order m_0 onto a cylinder of the form $(0, \rho_{kj}) \times \Delta_{kj}$, where Δ_{kj} is a domain in the variables $(\widetilde{\varphi}_{kj}, \tilde{y}_{kj})$.

Consider the set of functions $u(x)$ for which, for each $k = 1, 2, \dots, P_0$, there exist derivatives

$$D^{\alpha}_{\tilde{x}k} \left[D^{\beta^{(l,k)}}_{\overline{x}k} u(x) \right] \qquad (l = 1, \dots, \lambda_k;\ 0 \le |\alpha| \le m^{(l,k)})$$

in the sense of S. L. Sobolev. Let us write these derivatives, obtained for all k, l, and α, in the form of the sequence

$$D^{\alpha(1)}u,\ D^{\alpha(2)}u,\ \dots,\ D^{\alpha(Q)}u,\ \dots,\ D^{\alpha(R)}u \tag{1}$$

and suppose that

$$g^p(u) = \int_\Omega \sum_{\gamma=1}^{Q} b_\gamma(x) \left| D^{\alpha(\gamma)}u \right|^p \, dx < \infty \qquad (1 < p < \infty) \tag{2}$$

and, for every $k = 1, 2, \dots, P_0$,

$$g^p(u) = \sum_{l=1}^{\lambda_k} \int_\Omega b^{(l,k)}(\tilde{x}_k)\, a^{(l,k)}(\overline{x}_k) \sum_{|\alpha| = m^{(l,k)}} \left| D^{\alpha}_{\tilde{x}k} \left[D^{\beta^{(l,k)}}_{\overline{x}k} u(x) \right] \right|^p dx, \tag{3}$$

where $b_\gamma(x)$, $b^{(l,k)}(\tilde{x}_k)$, $a^{(l,k)}(\bar{x}_k) > 0$ are continuous functions in Ω, and, if $1 \le j \le A_k$ and j' is such that $s_k(j') > s_k(j)$, then for any $\rho_{kj'} > 0$ there is a $\rho^{(0)}_{kj} > 0$ such that on

$$T_{\rho^{(0)}_{kj}} \setminus \bigcup_{s_k(j') > s_k(j)} T_{\rho_{kj'}}$$

with certain constants $c_1(\rho_{kj'}) > 0$ and $c_2(\rho_{kj'}) > 0$ the inequality

$$c_1 b^{(l,k)}(x) \le \rho_{kj}^{\gamma(l,k,j)} \Lambda_{lkj}(\tilde{y}_{kj}, \tilde{\varphi}_{kj}) \le c_2 b^{(l,k)}(x)$$

is satisfied

$$(k = 1, 2, \dots, P_0;\ l = 1, 2, \dots, \lambda_k).$$

If Ω_k is unbounded, then outside a ball of sufficiently large radius

$$0 < \varepsilon_1 \le b^{(l,k)}(\tilde{x}_k) |\tilde{x}_k|^{-\gamma(l,k)} \le \varepsilon_2 < \infty.$$

The resulting set of functions will be denoted by $L^{(\alpha)}_{p,\mathbf{b}}(\Omega)$ ($\mathbf{b} = (b_1, \dots, b_Q)$, $\vec{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(R)})$).

Let

$$\tilde{\Phi}_{lkj} = m^{(l,k)} - 1 - \left[(\gamma^{(l,k,j)} + s_k(j)) : p \right]$$

for $j = 1, \dots, A_k$;

$$\tilde{\Phi}_{lk} = m^{(l,k)} - (\gamma^{(l,k)} + q_k) : p,$$

if there exists an r $(1 \leq r \leq m^{(l,k)})$ such that

$$\gamma^{(l,k)} - rp + 1 = 0;$$

otherwise

$$\tilde{\Phi}_{lk} = m^{(l,k)} - 1 - [(\gamma^{(l,k)} + q_k) : p].$$

Put

$$\Phi_{lkj} = \max(-1, \min(m^{(l,k)} - 1, \tilde{\Phi}_{lkj})), \quad \Phi_{lk} = \max(-1, \min(m^{(l,k)} - 1, \tilde{\Phi}_{lk})).$$

Definition. We shall say that $u(x) \in \overset{0}{L}{}^{(\alpha)}_{p,\mathbf{b}}(\Omega)$ if:

1) $u(x) \in L^{(\alpha)}_{p,\mathbf{b}}(\Omega)$;

2)

$$\lim_{\rho_{kj} \to 0} D^{\alpha}_{\tilde{x}_k} \left[D^{\beta(l,k)}_{\tilde{x}_k} u(\rho_{kj} \tilde{y}_{kj}, \tilde{\varphi}_{kj}, \bar{x}_k) \right] = 0 \quad (0 \leq |\alpha| \leq \Phi_{lkj})$$

for almost all $(\tilde{y}_{kj}, \tilde{\varphi}_{kj}, \bar{x}_k)$, and in the case when Ω_k is unbounded;

3)

$$\lim_{|\tilde{x}_k| \to \infty} D^{\alpha}_{\tilde{x}_k} \left[D^{\beta(l,k)}_{\tilde{x}_k} u(|\tilde{x}_k|, \tilde{\theta}_k, \tilde{x}_k) \right] = 0 \quad (\Phi_{lk} + 1 \leq |\alpha| \leq m^{(l,k)} - 1)$$

for almost all $(\tilde{\theta}_k, \bar{x}_k)$, where $\tilde{\theta}_k$ is a set of angular spherical coordinates in E_{q_k}

$$(k = 1, \dots, P_0;\ l = 1, \dots, \lambda_k;\ j = 1, \dots, A_k).$$

Theorem 1. Suppose that for all $k = 1, 2, \dots, P_0$, $l = 1, \dots, \lambda_k$ at least one of the following conditions is satisfied:

1. There exists a $j(l)$ such that $\Phi_{lkj} = m^{(l,k)} - 1$.
2. Ω_k is unbounded and $1 + \Phi_{lk} = 0$.
3. Ω_k is bounded and there exists a $j(l)$ such that $\Phi_{lkj} \geq \Phi_{lk}$.

Then $C^{(\infty)}_0(\Omega)$ is dense in $\overset{0}{L}{}^{(\alpha)}_{p,\mathbf{b}}(\Omega)$ in the seminorm $g(u)$.

II. Consider the system of equations

$$L_i(u) = \sum_{\gamma=1}^{R_i} (-1)^{|\alpha^{(i\gamma)}|} D^{\alpha^{(i,\gamma)}} \varphi_{i\gamma}(x, D^{\alpha(p,q)} u_p(x)) = f_i(x) \tag{4}$$

$(i;\ p = 1, 2, \dots, s;\ \gamma = 1, 2, \dots, R_i;\ q = 1, 2, \dots, R_p$; the functions $\varphi_{i\gamma}(x, y^{(i\gamma)}_{pq})$

$$(-\infty < y^{(i,\gamma)}_{pq} < \infty)$$

depend on $R = 1 + \sum_{i=1}^{s} R_i$ arguments and are continuously differentiable with respect to all of them, $u(x) = (u_1(x), \dots, u_s(x))$ is an unknown vector-function. By $\widetilde{H}$, H we denote the sets of vector-functions for which

$$u_i(x) \in L_{2,\mathbf{b}_i}^{(\vec{\alpha}_i)}(\Omega), \quad \overset{0}{L}{}_{2,\mathbf{b}_i}^{(\vec{\alpha}_i)}(\Omega)$$

respectively

$$(i = 1, 2, \dots, s, \quad \mathbf{b}_i(x) = (b_{i1}(x), \dots, b_{iQ_i}(x)), \quad \vec{\alpha}_i = (\alpha^{(i,1)}, \dots, \alpha^{(i,R_i)})).$$

In what follows, everywhere, the addition of the index i to notation encountered earlier will mean that it refers to the space of the i-th component of the vector $u(x)$. Below we shall assume that for any $i = 1, \dots, s$:

a) for some pair $(l_0^{(i)}, k_0^{(i)})$

$$|\beta^{(l_0^{(i)}, k_0^{(i)})}| = 0$$

and b) the condition of Theorem 1 is satisfied.

Let $\beta_{i\gamma}(x) > 0$ be such that

$$\int_\Omega \beta_{i\gamma}(x) \left| D^{\alpha(i,\gamma)} u_i(x) \right|^2 dx \le g_i^2(u_i) \le$$

$$\le \int_\Omega \sum_{\gamma=1}^{Q_i} b_{i\gamma}(x) \left| D^{\alpha(i,\gamma)} u_i(x) \right|^2 dx$$

for $u \in H$, and for $1 \le \gamma \le Q_i$ put $\beta_{i\gamma}(x) = b_{i\gamma}(x)$. The functions $\beta_{i\gamma}$ can also be obtained starting from the zero boundary conditions for the functions $u_i(x)$ and their derivatives and the known Hardy inequality ((5), p. 296). Here we note only that the indicated functions $\beta_{i\gamma}(x) > 0$ in Ω exist for all $i = 1, 2, \dots, s$ and $\gamma = 1, 2, \dots, R_i$.

Denote by H_0 the set of vector-functions all of whose components belong to $C_0^{(m_0)}(\Omega)$, where $m_0 \ge \max_{i\gamma} |\alpha^{(i\gamma)}|$. By $\hat{H}$ we denote the set of those $u(x) \in \widetilde{H}$ for which

$$\int_\Omega \beta_{i\gamma}^{-1}(x) \left| \varphi_{i\gamma}\left(x, D^{\alpha(p,q)} u_p(x)\right) \right|^2 dx < \infty.$$

Suppose that the following conditions are satisfied:

)

$$\sup_{y_{\rho\sigma}^{(i\gamma)}} \left| \frac{\partial \varphi_{i\gamma}}{\partial y_{pq}^{(i,\gamma)}} \left(x, y_{\rho\sigma}^{(i,\gamma)}\right) \right| \le \tilde{c}\, \beta_{i\gamma}(x) \beta_{pq}(x) \quad (1 \le \rho \le s;\ 1 \le \sigma \le R_\rho);$$

) there exists a matrix $M(x) = \|m_{\theta i}(x)\|$ $(1 \le \theta, i \le s)$, for which $m_{\theta i} \in C^{(m_0)}(\Omega)$ $(\theta, i = 1, 2, \dots, s)$ and $|M(x)| \ne 0$ in Ω; moreover, if we denote

$$v_i = \sum_{\theta=1}^{s} m_{\theta i} u_\theta(x) \quad (u(x) \in H_0),$$

and by $\mu_{\rho\sigma}^{(i,\gamma)}(x)$ the functions in the relation

$$D^{\alpha(i,\gamma)} v_i = \sum_{\rho\sigma} \mu_{\rho\sigma}^{(i,\gamma)} \cdot D^{\alpha(\rho,\sigma)} u_\rho,$$

which we shall regard as satisfied, then for all i, γ, ρ, σ

$$\left| \mu_{\rho\sigma}^{(i,\gamma)}(x) \right| \beta_{i\gamma}(x) \leq \tilde{c}\, \beta_{\rho\sigma}(x);$$

) there exists a system of functions $\{\Delta_{pq\rho\sigma}(x)\}$ such that

$$\sum_{pq\sigma} \int_\Omega \Delta_{pq\rho\sigma}(x) \times$$

$$\times D^{\alpha(p,q)} u_{pD}^{\alpha(p,\sigma)} u_\rho \, dx = 0$$

for every $u \in H_0$, and the functions

$$\tau_{pq\rho\sigma} = \sum_{i\gamma} \frac{\partial \varphi_{i\gamma}}{\partial y_{pq}^{(i,\gamma)}} \left(x, y_{\delta\omega}^{(i,\gamma)} \right) \left(y_{\rho\sigma}^{(i,\gamma)}(x) + \Delta_{pq\rho\sigma} \right) \quad (\delta = 1, \dots, s;\ \omega = 1, \dots, R_\delta)$$

are representable in the form

$$\tau_{pq\rho\sigma} = \tau_{pq\rho\sigma}^{(1)} + \tau_{pq\rho\sigma}^{(2)},$$

where

$$\sum_{pq\rho\sigma} \tau_{pq\rho\sigma}^{(1)} t_{pq} t_{\rho\sigma} \geq \sum_{i=1}^{s} \sum_{\gamma=1}^{R_i} b_{i\gamma}(x) t_{i\gamma}^2,$$

$$\left| \tau_{pq\rho\sigma}^{(2)} \right|^2 \leq \gamma_{pq\rho\sigma} \beta_{pq}(x) \beta_{\rho\sigma}(x),$$

if $(p, q) \neq (\rho, \sigma)$, and

$$-\tau_{pqpq}^{(2)} \leq \gamma_{pqpq} \beta_{pq}(x) \quad (p, \rho, i = 1, 2, \dots, s;\ q = 1, 2, \dots, R_p;\ \sigma = 1, 2, \dots, R_\rho;\ \gamma = 1, 2, \dots, R_i),$$

where $\gamma_{pq\rho\sigma}$ are absolute constants;

)

$$\sum_{pq\rho\sigma} \gamma_{pq\rho\sigma} < 1.$$

Theorem 2. Let $\hat{u}_0 \in \hat{H}$, $f(x) = (f_1(x), \dots, f_s(x)) \in H^*$, and suppose that for each $i = 1, \dots, s$ the conditions a)–e) listed above are satisfied. Then there exists a unique $u(x) \in \hat{H}$ (a generalized solution of system (4)) such that

$$\int_\Omega \sum_{i\gamma} \varphi_{i\gamma} \left(x, D^{\alpha(p,q)} u_p(x) \right) D^{\alpha(i\gamma)} v_i(x) \, dx = \int_\Omega \sum_{i=1}^{s} f_i(x) v_i(x) \, dx$$

for any $v \in H_0$, and $u - \hat{u}_0 \in \hat{H}$.

In the proof of Theorem 2, Theorem 3 of paper (3) was used. I express my deep gratitude to S. L. Sobolev and L. D. Kudryavtsev for their attention to the work and for discussion of the results.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.