F. A. BEREZIN

MATHEMATICS

1965

SovietRxiv

Abstract

Full Text

F. A. BEREZIN

ASYMPTOTICS OF THE EIGENFUNCTIONS OF THE MANY-PARTICLE SCHRÖDINGER EQUATION

(Presented by Academician I. G. Petrovskii, January 15, 1965)

MATHEMATICS

1. Consider a system of n identical one-dimensional particles with pair interaction. The Schrödinger equation of such a system has the form

$$\left\{-\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)+\sum_{i<j} v(x_i-x_j)\right\}\psi=k^2\psi. \tag{1}$$

The expression standing in braces is the energy operator of the system, which we shall denote by H_n. Throughout the article it is assumed that the potential $v(x)$ satisfies the conditions *

$$v(x)=\int_{\mu_0}^{\infty} e^{-\mu|x|}\xi(\mu)\,d\mu, \qquad \int_{\mu_0}^{\infty}|\xi(\mu)|\,d\mu<\infty, \qquad \mu_0>0. \tag{2}$$

Among the solutions of equation (1) there are functions symmetric and antisymmetric with respect to permutations of $x_1,\ldots,x_n$. In this paper the asymptotics of the symmetric and antisymmetric solutions of equation (1) is found.

2. The variables separate for the operator H_2, and therefore there exists a complete system of eigenfunctions of the form

$$\psi(x_1,x_2\mid p_1,p_2)=e^{\frac{i}{2}(p_1+p_2)(x_1+x_2)}\widetilde{\psi}\left(\frac{x_1-x_2}{\sqrt{2}}\,\middle|\,\frac{p_1-p_2}{\sqrt{2}}\right), \tag{3}$$

where $\widetilde{\psi}(x\mid p)$ is an eigenfunction of the Sturm-Liouville operator $-d^2/dx^2+v(x)$ with eigenvalue equal to p^2.

The symmetric eigenfunctions of the operator H_2 of the form (3) constitute a complete system in the space of symmetric functions of two variables with summable square, and the antisymmetric ones—in the space of antisymmetric functions.

Using known results on the Sturm-Liouville operator (see, for example, ([1],[2])), it is not difficult to show that the symmetric and antisymmetric eigenfunctions of the form (3) can be normalized so that inside the angle $x_1 - x_2 > \varepsilon|x_1 + x_2|$ ($\varepsilon > 0$) they have the form

$$\psi_s(x_1, x_2 \mid p_1, p_2) = c_s(p_1 - p_2)e^{i(p_1x_1+p_2x_2)} + c_s(p_2 - p_1)e^{i(p_2x_1+p_1x_2)} + o(1),$$
$$\psi_a(x_1, x_2 \mid p_1, p_2) = c_a(p_1 - p_2)e^{i(p_1x_1+p_2x_2)} - c_a(p_2 - p_1)e^{i(p_2x_1+p_1x_2)} + o(1), \tag{4}$$

where $c_s(p) = c_s(-p)$, $c_a(p) = -c_a(-p)$. The eigenvalue in both cases is equal to $p^2 = p_1^2 + p_2^2$.

---

* Probably these conditions are more restrictive than is necessary for the validity of the theorems stated below (see in this connection ([4]), where the case $v(x) = \lambda\delta(x)$ is considered).

Theorem 1. Denote by Γ the polyhedral angle $x_1 > x_2 > \dots > x_n$, and by Γ_1 an arbitrary closed polyhedral angle with vertex at the origin, all points of which, except the vertex, lie strictly inside Γ. There exist symmetric ψ_s and antisymmetric ψ_a solutions of equation (1), having inside Γ_1 the form:

$$\psi_s(x \mid p) = \prod_{\alpha<\beta} c_s(p_\alpha - p_\beta)e^{i(p_1x_1+\dots+p_nx n)} + \dots + O\left(\frac{1}{\sqrt{|x|}}\right), \tag{5_s}$$

$$\psi_a(x \mid p) = \prod_{\alpha<\beta} c_a(p_\alpha - p_\beta)e^{i(p_1x_1+\dots+p_nx n)} + \dots + O\left(\frac{1}{\sqrt{|x|}}\right). \tag{5_a}$$

The functions c_s, c_a here are the same as in (4); the dots replace the terms obtained from the one written by all possible permutations of $p_1, \dots, p_n$, where in (5_s) all terms enter with a plus sign, while in (5_a) with a plus sign if the permutation is even and with a minus sign otherwise. The eigenvalue in both cases is equal to $p^2 = p_1^2 + \dots + p_n^2$.

Theorem 2. In the case where the operator $-d^2/dx^2 + v(x)$ has no discrete spectrum, the functions (5_s), (5_a) form complete systems in the spaces consisting respectively of symmetric or antisymmetric square-summable functions of n variables.

For brevity, introduce the subscript ν, taking the values s and a. Consider the symmetric and antisymmetric eigenfunctions of scattering theory:

$$\psi_\nu^-(x \mid p) = \lim_{\varepsilon\to+0} i\varepsilon(H_n - p^2 - i\varepsilon)^{-1}\varphi_{\nu 0}(x \mid p), \tag{6}$$

where

$$\psi_{s0}(x \mid p) = \sum \exp\{i(p_{k_1}x_1 + \dots + p_{k_n}x_n)\},$$

$$\psi_{a0}(x \mid p) = \sum \pm \exp\{i(p_{k_1}x_1 + \dots + p_{k_n}x_n)\}.$$

In both cases the sums are over all permutations; in the second case the plus or minus sign is put according to the parity of the permutation.

Theorem 3. The functions ψ_{s-}, ψ_{a-}, for $x \in \Gamma_1$, are connected with the functions (5_s), (5_a) by the relations $(\nu = s, a)$:

$$\psi_{\nu-}(x \mid p) = \left[\prod_{\alpha<\beta} \bar{c}_\nu(|p_\alpha - p_\beta|)\right]^{-1} \psi_\nu(x \mid p) + O\left(\frac{1}{\sqrt{|x|}}\right). \tag{7}$$

Denote by $G(x, y \mid E)$ the kernel of the operator $(H_n - E - i0)^{-1}$. The following theorem establishes an important property of the function G.

Theorem 4. Let $f(x_1, \dots, x_n)$ be a generalized function of the form

$$f = \delta(x_i - x_j)\varphi(x_1, \dots, x_n),$$

where φ is a classical function, constant on the hyperplanes $\sum a_{\alpha\beta}x_\beta = 0$, $1 \le \alpha \le k$, such that

$$\int |\varphi| \prod_\alpha \delta\left(\sum_\beta a_{\alpha\beta}x_\beta\right) dx_1, \dots, dx_n < \infty.$$

Then, for $x \in \Gamma_1$, the estimate holds

$$\int (L_{yG}(x, y))f(y)d^{ny} = O\big(|x|^{-(n-k-1)/2}\big), \tag{8}$$

where L_y is an arbitrary differential operator of order at most one with constant coefficients, acting on G as on a function of y.

In this theorem and in the preceding one, Γ_1 is the same polyhedral angle as in Theorem 1.

3. We present the idea of the proof of Theorems 1 and 3. Consider, inside the polyhedral angle Γ_n defined by the inequalities $x_1 > x_2 > \dots > x_n$, a solution of equation (1) of the special form

$$\varphi_n(x \mid p) = e^{i(p,x)}\left(1 + \int_{\Gamma_n} B_n(\mu \mid p)\,\delta(\mu_1 + \dots + \mu_n)e^{-(\mu,x)}d^n\mu\right). \tag{9}$$

The integral in (9) is extended to the polyhedral angle Γ'_n, dual to $\Gamma_n : \mu \in \Gamma'_n$, if $(\mu, x) \ge 0$ for $x \in \Gamma_n$. The eigenvalue to which φ_n belongs is equal to $p^2 = p_1^2 + \dots + p_n^2$. From this point on we shall assume that $p_1 + \dots + p_n = 0$ (the general case is easily reduced to this). The function φ_n need not be extendable beyond Γ_n, and therefore we shall call it a local solution. The function $B_n(\mu \mid p)$, which determines the local solution, is naturally sought in the form

$$B_n(\mu \mid p) = \sum_{k=2}^{n} \sum A_k(\mu_{i_1} \dots \mu_{i_k} \mid p_{i_1}, \dots, p_{i_k}) \prod_{i_1 \dots i_k} \delta(\mu_1) \dots \delta(\mu_n), \tag{10}$$

where $A_k(\mu_1 \dots \mu_k \mid p_1 \dots p_k)$ are functions equal to zero if the vector $\mu = (\mu_1, \dots, \mu_k)$ does not belong to Γ'_k; $\prod_{i_1 \dots i_k} \delta(\mu_1) \dots \delta(\mu_n)$ denotes the product of the functions $\delta(\mu_i)$ over all i from $i = 1$ to $i = n$, except $i = i_1, \dots, i_k$; the inner sum in (10) is taken over all permutations. For $k < n$ the functions A_k are the same as those occurring in the expression for $B_{n'}$, with $n' < n$.

Substituting φ_n into equation (1) and using the special form of the potential $v(x)$, after simple transformations we obtain for the functions A_k integral equations close to Volterra equations. Like ordinary Volterra equations, these equations are uniquely solvable. From their analysis we obtain the following important properties of the functions A_k: 1) $A_n(\mu \mid p) = 0$ inside the polyhedral angle $\tilde{\Gamma}'_n \subset \Gamma'_n$, whose sides are parallel to the sides of Γ'_n, and whose vertex lies strictly inside Γ'_n; 2) $A_n(\mu \mid p)$, as a function of p, admits analytic continuation into the domain $\operatorname{Im} p \in \Gamma_n$ and, for fixed p in this domain, is bounded with respect to μ.

Let us now consider the symmetric function $\Phi_n(x \mid p)$, defined inside Γ_n by the formula

$$\Phi_n(x \mid p) = \prod_{\alpha<\beta} c_s(p_\alpha - p_\beta)\, \varphi_n(x_1, \dots, x_n \mid p_1, \dots, p_n) + \cdots, \tag{11}$$

where the ellipsis replaces the sum of terms obtained from the one written by all possible permutations of $p_1, \dots, p_n$. The function (11) is continuous in the whole space and everywhere, except on the planes $x_\alpha = x_\beta$, satisfies equation (1). The jump of the normal derivative on the plane $x_\alpha = x_\beta$ will be denoted by $f_{\alpha\beta}(x)$; $f_{\alpha\beta}(x)$ turns out to be a linear combination of the functions

$$f_{\alpha\beta|k} = \int [i(p_\alpha + i\mu_\alpha) - i(p_\beta + i\mu_\beta)] A_k(\mu \mid p) e^{-(\mu,x)} \times \delta(\mu_1 + \cdots + \mu_k) \delta(\mu_{k+1}) \dots \delta(\mu_n)\, d^n\mu \tag{12}$$

and of the functions obtained from (12) by permutations of $p_1, \dots, p_n$ and by those permutations of x_i for which $x_{i_1} > x_{i_2} > \dots > x_{i_k}$.

The function $\Phi_n(x \mid p)$ is constructed with the following calculation, so that in the formation of $f_{\alpha\beta}$ the terms $f_{\alpha\beta|k}$ enter only for $k \geq 3$. We note that the function $f_{\alpha\beta|k}$ depends on $k-1$ variables: in formula (12) there participate k variables; moreover, according to (12), $f_{\alpha\beta|k}(x_1 + h, \dots, x_k + h) = f_{\alpha\beta|k}(x_1, \dots, x_k)$.

This property reduces the number of variables on which $f_{\alpha\beta|k}$ actually depends to $k-1$. The value of $f_{\alpha\beta|k}$ on the hyperplane $x_\alpha = x_\beta$ is, therefore, a function of $k-2$ variables. It is obvious that $f_{\alpha\beta|k}$ decreases exponentially when $x_1 + \dots + x_k = 0$ and $x_1 \geq x_2 \geq \dots \geq x_k$.

We shall now seek the eigenfunction in the form $\psi = \Phi_n + \varepsilon$. Substituting ψ into equation (1), we find for ε the equation

$$(H_n - p^2)\varepsilon = \sum_{\alpha<\beta} \delta(x_\alpha - x_\beta) f_{\alpha\beta}(x). \tag{13}$$

One solution of this equation is $\varepsilon = (H_n - p^2 - i0)^{-1} g$, where g denotes the right-hand side of (13). Thus, $\varepsilon(x) = \int G(x, y \mid p^2) g(y)\, d^n y$, and according to (8) ($L_y = 1$), $\varepsilon = O(|x|^{-1/2})$ inside Γ_1.

We note that the function $f_{\alpha\beta\gamma k}$ gives a contribution to ε of order $|x|^{-(k-2)/2}$. The antisymmetric case is considered analogously.

We give the idea of the proof of Theorem 3. For simplicity we restrict ourselves to the symmetric case and $n = 3$. Expand the function $\psi_0 = e^{i(p_1 x_1 + \dots + p_3 x_3)} + \cdots$ in the functions $\left(\prod_{i>j} \bar{c}(|p_i - p_j|)\right)^{-1} \psi_s(x \mid p)$:

$$\psi_0(x \mid p) = \int a(p, q) \left(\prod_{i<j} \bar{c}(|q_i - q_j|) \right)^{-1} \psi_s(x \mid q) \delta(q_1 + q_2 + q_3)\, d^3 q. \tag{14}$$

From (14) and (7) we find that

$$\psi_{s-}(x \mid p) = \int \lim_{\varepsilon \to +0} i\varepsilon \frac{a(p, q)}{q^2 - p^2 - i\varepsilon} \frac{\psi_s(x \mid q)}{\prod_{i<j} \bar{c}(|q_i - q_j|)} \delta(q_1 + q_2 + q_3)\, d^3 q. \tag{15}$$

Starting from (14) and (5_s), we obtain for $a(p, q)$ the expression

$$a(p, q) = \frac{1}{(2\pi i)^2} \frac{1}{\prod_{i<j} c(|q_i - q_j|)} \left\{ \frac{\theta(p_1 - p_2)\theta(p_2 - p_3)}{(p_1 - q_1 - i0)(p_3 - q_3 + i0)} \prod_{i<j} c(q_i - q_j) + \cdots \right\} + $$
$$+ \alpha(p, q), \tag{16}$$

where $\theta(p) = 1$ for $p > 0$ and $\theta(p) = 0$ for $p < 0$; the ellipsis replaces the terms obtained from the written one by symmetrization with respect to p_α and q_β; $\alpha(p, q)$ is the Fourier transform of the remainder term:

$$\alpha(p,q) = \frac{\alpha_1(p,q)}{p^2 - q^2 - i0} + \delta(p^2 - q^2)\alpha_2(p,q) + \alpha_3(p,q), \tag{17}$$

$\alpha_i(p,q)$ are classical bounded functions.

The identity holds (see (3))

$$\lim_{\varepsilon \to +0} i\varepsilon \frac{\delta(p_1 + p_2 + p_3 - q_1 - q_2 - q_3)}{(p_1^2 + p_2^2 + p_3^2 - q_1^2 - q_2^2 - q_3^2 + i\varepsilon)(p_1 - q_1 - i0)(p_3 - q_3 + i0)} =$$

$$= (2\pi i)^2 \theta(p_1 - p_2)\theta(p_2 - p_3)\delta(p_1 - q_1)\delta(p_2 - q_2)\delta(p_3 - q_3).$$

Using this formula and formula (15), we find that

$$\psi_{s-}(x \mid p) = \frac{\psi_s(x \mid p)}{\prod_{i<j} \bar{c}(|p_i - p_j|)} +$$

$$+ \int \delta(p^2 - q^2)\alpha_2(p,q)\delta(q_1 + q_2 + q_3)\, d^3q = \frac{\psi_s(x \mid p)}{\prod_{i<j} \bar{c}(|p_i - p_j|)} + O(|x|^{-1/2}).$$

Finally, we derive an equation whose investigation makes it possible to obtain an estimate of the kernel of the operator $G_n = (H_n - p^2 - i0)^{-1}$. For simplicity we restrict ourselves to the case $n = 3$ and omit the index n. Put $G = G_0 + G_0 T G_0$, where G_0 is the operator obtained from G for $v(x) = 0$. The operator T satisfies the equation

$$(E - K_{12} - K_{13} - K_{23})T = V_{12} + V_{13} + V_{23} = T_0,$$

where V_{ij} is the operator of multiplication by $V(x_i - x_j)$, and $K_{ij} = V_{ij} G_0$. Put $(E - K_{ij})^{-1} = E + L_{ij}$. Multiplying both sides of the equation for T by $(E + L_{12})(E + L_{13})(E + L_{23})$ and taking into account that

$$L_{ij} - K_{ij} - L_{ij}K_{ij} = (E + L_{ij})(E - K_{ij}) - E = 0,$$

we find the final equation

$$T = (E + L_{12})(E + L_{13})(E + L_{23})T_0 + RT.$$

Received
6 I 1965

CITED LITERATURE

1. Z. S. Agranovich, V. A. Marchenko, *The Inverse Problem of Scattering Theory*, Kharkov, 1960.
2. L. D. Faddeev, UMN, **14**, 4, 57 (1959).
3. F. A. Berezin, V. P. Sushko, ZhETF, **48**, no. 5 (1965).
4. F. A. Berezin, G. P. Pokhil, V. M. Finkelberg, *Vestn. Mosk. Univ.*, no. 1, 21 (1964).

Note: Figure translations are in progress. See original paper for figures.