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Abstract

Full Text

MATHEMATICS

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ON PONOMAREV' S THEORY OF ABSOLUTES OF TOPOLOGICAL SPACES*

1. V. Ponomarev defined [1–4] the absolute of a paracompact space as the limit of a zero-dimensional projection spectrum**—the "complete loosening" of the maximal spectrum of the given paracompact space, and I regard this construction of the absolute as especially visual and natural. It therefore seemed interesting to me to give (and this is the subject of the present note) a direct definition of a class of zero-dimensional spectra, which I call "absolute" spectra, a class sufficiently broad that, as the limit spaces of the spectra belonging to it, one obtains all Ponomarev absolutes. At the same time, the chief merit of absolute spectra is that they stand in a one-to-one correspondence with the spaces determined by them.

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* In connection with the history of the question I consider it necessary to note the following. The first actual construction of the absolute without any applications of it is contained in the classic memoir [7] of M. Stone, published as early as 1937. In the work [8] of A. M. Gleason, published in 1959, M. Stone' s construction is considered anew for bicompacts (after it had already figured, as an auxiliary device, in the work [9] of S. V. Fomin). Gleason, in proving the projectivity of the absolute, thereby in fact also establishes the property of the absolute of generating, by its homeomorphisms, all one-to-one irreducible continuous (respectively, perfect) mappings of bicompacts (respectively, locally bicompact spaces).

V. I. Ponomarev not only constructs—and moreover by a completely new and "spectral" method—the absolute for arbitrary paracompacts, but also, for arbitrary paracompacts, proves by the same spectral method the main theorem:

The homeomorphism of the absolutes of two spaces is equivalent to the existence of an irreducible perfect—generally speaking many-valued—mapping of one of these spaces onto the other: the formula

$$f = \pi_y h \pi_x^{-1} \tag{1}$$

contains all and only such mappings.

This result (which could hardly have been formulated without the preceding works of the same author on many-valued mappings) is derived by V. Ponomarev from the spectral construction he gives, first, of the paracompacts themselves, second, of all their perfect preimages, in particular of their absolutes, and third, of their irreducible perfect mappings. All this, as a whole, is contained in the "Ponomarev theory of absolutes," which I regard as one of the most outstanding achievements of set-theoretic topology in recent decades. Like every significant theory, it has its predecessors, and they do not diminish it: the theory of A-sets, created by M. Ya. Suslin in 1917, did not become less significant because the A-operation had been constructed two years earlier, and the first A-sets, which were not B-sets, had been discovered by Lebesgue as early as 1905. Likewise, the theory of nowhere bicompact extensions created by E. Čech in 1936 does not lessen its significance because Čech' s construction, in its full generality, was given by A. N. Tikhonov as early as 1925 (and its countable special case was used by P. S. Uryson even earlier—in 1922–1923).

Only V. Ponomarev' s works drew the attention of new researchers to absolutes; after these works, and referring to them, various authors—and earlier than all, and most interestingly, the young Moscow topologist S. Iliadis [(6)] (a student of Yu. M. Smirnov)—gave quite different constructions of the absolute and extended them to still broader classes of spaces (up to all Hausdorff spaces inclusive). Now, after the work of S. Iliadis, V. Ponomarev has shown that the same degree of generality can be attained by his (i.e., spectral) methods [(4)], first expounded by him as early as 1960 in [(10)].

** All definitions concerning projection spectra may be found by the reader in [(5)], and also in [(3)]; I follow throughout the terminology of these works.

Absolute spectra are obtained by applying to a special case of zero-dimensional spectra the theory developed in [5].

2. A zero-dimensional, i.e. consisting of zero-dimensional complexes, projection spectrum $S = \{|\alpha|, \mathfrak{D}_{\alpha}^{\alpha'}\}$ will be called absolute if it satisfies the following conditions:

1^0. The spectrum S is complete*.

2^0. The spectrum S has no order strengthening** distinct from it.

3^0. The spectrum S has no equivalent indices (i.e. indices α, α' such that the projection $\mathfrak{D}_{\alpha}^{\alpha'}$ would be a one-to-one mapping of the complex $|\alpha'|$ onto the complex $|\alpha|$.

4^0. The spectrum S is not a subspectrum*** of any spectrum distinct from it that satisfies conditions 1^0–3^0.

Theorem 1. *If a space X_0 is an absolute of some paracompactum X (and, consequently, its own absolute), then it is the space of some (and moreover unique) absolute spectrum.*

We shall prove that this spectrum is the maximal spectrum of the space X_0, i.e. the spectrum corresponding to the system of all decompositions of the paracompactum X_0. We shall call it the **absolute spectrum** (of every) paracompactum X having the given absolute X_0.

Since X_0, being an absolute, is an extremally disconnected space [4], every decomposition of the space X_0 is a disjoint covering.

Consequently, the maximal spectrum S_0 of the space X_0 is a zero-dimensional spectrum, and it (by virtue of the results of [5]) satisfies conditions 1^0–4^0, i.e. is an absolute spectrum.

To prove that S_0 is the unique absolute spectrum with the same space X_0, let us note that the following holds.

Lemma 1. *Every complete zero-dimensional spectrum is canonical (in the sense of [5]).*

Indeed, let the zero-dimensional spectrum $S = \{|\alpha|, \mathfrak{D}_\alpha^{\alpha'}\}$ be complete, and let $e_\alpha \in |\alpha|$, $e_{\alpha'} \in \alpha'$, $\alpha' > \alpha$, $\mathfrak{D}_\alpha^{\alpha'} e_{\alpha'} = e_\alpha$. Take a thread containing the vertex $e_{\alpha'}$; it also contains e_α and (as every thread of a zero-dimensional spectrum) is maximal—the condition of canonicity is satisfied.

Thus, an absolute spectrum, being canonical, is a maximally perfect spectrum in the sense of [5]; and then its uniqueness follows from Theorem IV of [5] (p. 130).

We shall now denote the zero-dimensional complexes $|\alpha|$ simply by α.

Theorem 2. *The space $\breve{S}_0$ of the absolute spectrum $S_0 = \{\alpha, \mathfrak{D}_\alpha^{\alpha'}\}$ is a (completely regular space) that is its own absolute.*

* This means that for any simplex $t_\alpha \in |\alpha|$ of any complex $|\alpha|$ of the spectrum S and for any index $\alpha' > \alpha$ there is a simplex $t_{\alpha'} \in |\alpha'|$ projecting onto t_α:

$$\mathfrak{D}_\alpha^{\alpha'} t_{\alpha'} = t_\alpha.$$

Completeness of the spectrum S means that, without changing the space of the spectrum, one cannot discard from the spectrum S even one simplex t_α, or that for every simplex t_α there is a thread containing this simplex.

** We say that a directed set $\mathfrak{A}'$ is an order strengthening of the directed set $\mathfrak{A}$ if $\mathfrak{A}$ and $\mathfrak{A}'$ consist of the same elements, and from $\beta > \alpha$ in $\mathfrak{A}$ it follows that $\beta > \alpha$ also in $\mathfrak{A}'$.

A spectrum S' is called an order strengthening of the spectrum S if the set of indices $\mathfrak{A}'$ of the spectrum S' is an order strengthening of the set of indices $\mathfrak{A}$ of the spectrum S, and both spectra consist of one and the same set of complexes, and for $\beta > \alpha$ the projections $\mathfrak{D}_\alpha^\beta$ are the same in both spectra.

*** A spectrum S' is called a subspectrum of the spectrum S if the directed set of indices of the spectrum S' is a cofinal part of the set of indices of the spectrum

S, and the elements of both spectra (complexes and projections) having the same indices are identical.

Lemma 2. *Every complete spectrum* Σ *containing the spectrum* S_0 *as a subspectrum is zero-dimensional.*

Indeed, let $\tau^n = |a_0 \dots a_n|$ be some simplex of some complex $|\alpha^*| \in \Sigma$. Take the index α (of the spectrum S_0) following α^* in Σ. Then (by completeness of the spectrum Σ) there must exist a simplex $t_\alpha = e_\alpha \in |\alpha| = \alpha$, projecting onto τ^n, which is possible only in the case $n = 0$. Since every perfect spectrum (in the sense of (5)) is complete, every perfect spectrum Σ containing the spectrum S_0 as a subspectrum is necessarily a zero-dimensional spectrum satisfying conditions 1°–3°, and therefore (by condition 4°, which the spectrum S_0 satisfies) the spectrum Σ must coincide with S_0.

In other words, the absolute spectrum S_0 is a maximal perfect spectrum (in the sense of the paper (5)), and therefore it is the maximal spectrum (i.e. the spectrum of the system of all refinements) of its space $\widetilde{S}_0$. Since it is at the same time zero-dimensional, hence regular, the space $\widetilde{S}_0$ is a regular space every refinement of which is disjoint. In particular, every cover of the form $\{[\Gamma], \widetilde{S}_0 \setminus \Gamma\}$ is disjoint, where Γ is any canonical open set of the space $\widetilde{S}_0$; hence $[\Gamma]$ is open and $\widetilde{S}_0$ is extremely disconnected (see (4)) and completely regular, as was required to prove.

3. When is the space $\widetilde{S}_0$ of the absolute spectrum S_0 paracompact?

The answer to this question is contained in the paper (3), Ch. 1, § 2: the space $\widetilde{S}_0$ of the spectrum S_0 is paracompact if and only if the spectrum S_0 is uniform in the sense of Ponomarev (3). As applied to absolute spectra S_0, uniformity means the following.

We shall call a cover of the spectrum S_0 any set $\psi = \{e\}$ of vertices (of complexes $\alpha \in S_0$) having the property that every thread of the spectrum S_0 contains at least one vertex $e \in \psi$.

An absolute spectrum is uniform if for every one of its covers ψ there exists an α such that every vertex of the complex α either itself is an element of the cover ψ, or is projected into a vertex that is an element of ψ.

4. All paracompacts are spaces of spectra obtained by strengthening (in the sense of Ponomarev, (3), Ch. 3, § 3, p. 112) absolute spectra* (the process of strengthening consists in the fact that on the set of vertices forming the given zero-dimensional complex α, a complex $|\alpha|$ is constructed—in general, already not zero-dimensional—and in such a way that the projections defined in S_0 determine simplicial mappings $\mathfrak{F}_\alpha^{\alpha'}$ of the obtained complexes $|\alpha|$, so that a new projection spectrum S is obtained, having the spectrum S_0 as its "complete weakening"). If the spectrum S, which is a strengthening of the uniform absolute spectrum S_0, has as its space a Hausdorff space $\widetilde{S}$, then $\widetilde{S}$ is a paracompact (the property of uniformity is preserved under

strengthening). However, as a result of strengthening absolute spectra, one may also obtain spectra whose spaces are non-Hausdorff T_1-spaces. It would be interesting to find out what these spaces are; in any case they are "θ-perfect" (in the sense of S. V. Fomin) images of completely regular extremely disconnected spaces (absolutes).

5. In the bicompact case the whole picture is simplified: the absolute of any bicompact is the space of a uniquely determined finite (i.e. consisting of finite complexes) absolute spectrum; conversely, the space $\widetilde{S}_0$ of every absolute finite spectrum S_0 is a bicompact that is its own absolute.

* These absolute spectra automatically turn out to be uniform, which follows from the maximality property of absolute spectra (property 4°) and Ponomarev' s arguments ((3), p. 104).

Thus, to each bicompactum there corresponds uniquely its (finite) absolute spectrum. As spaces of Ponomarev extensions of finite absolute spectra one obtains all bicompacta and, possibly, some non-Hausdorff bicompact T_1-spaces; the question is, which ones exactly?

6. Let us return in conclusion to the general case of arbitrary paracompacta. The remarkable theorem of V. I. Ponomarev, which reduces all (even multivalued) irreducible perfect mappings of one paracompactum onto another to homeomorphisms between their absolutes, receives the following interpretation in terms of absolute spectra: regarding the given homeomorphism between the absolutes $WX = \dot{X}$ and $WY = \dot{Y}$ as a certain identification of the absolute spectra of the paracompacta X and Y, we consider as corresponding to one another* any two maximal threads (maximal spectra** of X and Y, respectively) that embrace one and the same thread of the (common) absolute spectrum of both spaces X and Y. If the identification of the absolute spectra of X and Y (corresponding to the given homeomorphism of the absolutes) is such that each maximal thread of the spectrum X is embraced by a unique maximal thread of the spectrum Y—and only in this case—we obtain a single-valued perfect mapping of the paracompactum X onto Y.

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* Under the (multivalued) mapping $X \leftrightarrow Y$, defined by the given homeomorphism between the absolutes $\dot{X}$ and $\dot{Y}$, i.e. by the given identification of the absolute spectra of X and Y.

** These spectra are extensions of the absolute spectra of the paracompacta X and Y identified with one another.

Note: Figure translations are in progress. See original paper for figures.