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Abstract

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ON THE EIGENFUNCTIONS OF AN ELLIPTIC OPERATOR IN CERTAIN UNBOUNDED DOMAINS

(Presented by Academician I. G. Petrovskii, 29 X 1964)

Let, in a domain D of n-dimensional Euclidean space, there be defined an elliptic differential operator

$$\mathcal{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + b(x)u,$$

where $x = (x_1, x_2, \ldots, x_n)$; the matrix $\|a_{ij}\|$ is symmetric and positive definite; the coefficients of the operator are real; $a_{ij}(x) \in C_1(\overline{D})$, $b(x) \in C(\overline{D})$. The domain D is the union of a bounded domain D_0, for which $x_1 < 0$, and the cylinder $\{[0,\infty) \times \Omega\}$, where Ω is a bounded domain in the space $(x_2, x_3, \ldots, x_n)$. The boundary $S \in C_2$, and for $x_1 > 0$ the operator $\mathcal{L}u \equiv \Delta u$.

It is known (^{1}) that, in order to single out a unique solution of the problem

$$\mathcal{L}u + \lambda u = f, \tag{1}$$

$$u|_S = 0 \tag{2}$$

for positive real λ, one must require the fulfillment of "partial radiation conditions" :

$$\frac{dw_k}{dx_1} - i\sqrt{\lambda - \alpha_k}\, w_k \xrightarrow[x_1\to\infty]{} 0 \quad \text{for } \alpha_k < \lambda, \tag{3}$$

$$w_k \xrightarrow[x_1\to\infty]{} 0 \quad \text{for } \alpha_k > \lambda,$$

where

$$w_k(x_1) = \int_\Omega u(x)\varphi_k(x^*)\,dx^*.$$

Here $x^* = (x_2, x_3, \dots, x_n)$; $\varphi_k(x^*)$ are eigenfunctions; α_k are the eigenvalues of the Dirichlet problem for the operator $-\Delta$ in the domain Ω. If the homogeneous equation

$$\mathcal{L}u + \lambda u = 0 \tag{4}$$

has no nontrivial solution in $W_2^2(D)$ equal to zero on the boundary, then for a finite function $f(x) \in \mathcal{L}_2$ the problem (1), (2) has a unique solution $u(x) \in W_2^2$ in every finite part of D and satisfying conditions (3) (^{2})*.

In the present paper we investigate the eigenvalues of the operator $-\mathcal{L}$, i.e., those λ for which equation (4) has a nontrivial solution in $W_2^2(D)$ equal to zero on the boundary. Of special interest are those eigenvalues which are greater than α_1, since the remaining points of the half-line $[\alpha_1, \infty)$ are points of the continuous spectrum of the operator under consideration. Its eigenvalues are of finite multiplicity and have no finite limit points (^{3}).

For a certain class of domains it has been proved that the Laplace operator has no eigenvalues for them (^{2}). As an example of an operator whose point spectrum lies on the continuous spectrum, one may take the operator

$$Mu \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + a(x_1)u$$

in the half-strip $[0, \infty) \times [0, \pi]$ on functions equal to zero on the boundary. If $z(x_1)$ is a solution of the equation $z'' + a(x_1)z - \mu^2 z = 0$ such that $z(0) = z(\infty) = 0$ (the corresponding real finite function $a(x_1)$ is easy to construct), then $k^2 - \mu^2$ are eigenvalues, and $z(x_1)\sin kx_2$ are eigenfunctions of the operator $-M$ for all natural k. One can construct domains where the operator $-\Delta$ also has eigenvalues on the continuous spectrum. Moreover, for some domains there is, in addition, a discrete spectrum (to the left of the point a_1), while for other domains it is absent.

Let $\lambda \neq a_k$ for any k, and let $\beta > 0$ be a number smaller than

$$\min_k \sqrt{|\lambda - a_k|}.$$

* The result given in (^{2}) differs inessentially from the one stated above because of other restrictions imposed

Denote by $H_{2,\beta}$ the Hilbert space of functions with scalar product

$$(uv)_{2,\beta} = \int_D e^{2\beta x_1} \left(u\overline{v} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \overline{v}}{\partial x_i \partial x_j} \right) dx,$$

and $H_{0,\beta}$ the space with scalar product

$$(u, v)_{0,\beta} = \int_D e^{2\beta x_1} u\overline{v}\, dx.$$

Theorem 1. Let the function $f(x) \in H_{0,\beta}$. In order that problem (1), (2) have a solution

$$u(x) = \tilde{u}(x) + \sum_{a_k < \lambda} c_k \exp\left[i\sqrt{\lambda - a_k}\, x_1\right] \varphi_k(x^*), \tag{5}$$

where $\tilde{u}(x) \in H_{2,\beta}$, it is necessary and sufficient that

$$\int_D f(x)v(x)\, dx = 0 \tag{6}$$

for every solution $v(x) \in H_{2,\beta}$ of problem (4), (2).

For the proof, consider the domains $D_\rho = D \cap (x_1 < \rho)$ for $\rho > 0$. Denote $S_\rho = S \cap (x_1 < \rho)$. Choose $R > 0$ such that in the domains D_R and $D_R \setminus D_0$ there exists a unique solution of the Dirichlet problem for equation (1). Denote by B the space of functions $\varphi(x^*) \in W_2^{3/2}(\Omega)$ that are zero on the boundary of Ω (see (^{4})). Let $\varphi(x^*) \in B$. Then in the domain $D \setminus D_0$ there exists a unique solution of the equation $\Delta u + \lambda u = 0$ of the form (5) and such that $u|_{S \setminus S_0} = 0$, $u(0, x^*) = \varphi(x^*)$. In this case $u(R, x^*) \in B$, and the operator A_1, taking the function $\varphi(x^*)$ to $u(R, x^*)$, is completely continuous. If $f(x) \in H_{0,\beta}$, then in the domain $D \setminus D_0$ there exists a unique solution of the equation $\Delta u + \lambda u = 0$ of the form (5), equal to zero on the boundary of $D \setminus D_0$. In this case the operator A_2, taking $f(x) \in H_{0,\beta}$ to $u(R, x^*) \in B$, is bounded. If $u(x)$ is a solution of equation (1) in the domain D_R, equal to zero on the boundary, then $u(0, x^*) \in B$, and the operator A_3, taking $f(x) \in H_{0,\beta}$ to $u(0, x^*)$, is bounded. Finally, if $u(x)$ is a solution of equation (5) in the domain D_R, which is equal to zero on S_R and equal to $\psi(x^*)$ for $x = R$, then $u(0, x^*) \in B$, and the operator A_4, taking $\psi(x^*)$ to $u(0, x^*)$, is bounded.

Thus, let $u(x)$ be a solution of problem (1), (2) of the form (5). Denote $u(R, x^*)$ by $\psi(x^*) \in B$. Then from the preceding it follows that the function $\psi(x^*)$ satisfies the equation

$$\psi = A_2 f + A_1(A_3 f + A_4 \psi) \tag{7}$$

or

$$\psi = A_5 \psi + A_6 f,$$

where the operator $A_5 = A_1 A_4$ is completely continuous in B, and the operator $A_6 = A_2 + A_1 A_3$ takes $H_{0,\beta}$ into B and is bounded.

Equation (7) and problem (1), (2) are equivalent. The Fredholm theorems are valid for equation (7). On the other hand, if problem (1), (2) has a solution, then relation (6) is fulfilled. From this it is easy to derive the conclusion of Theorem 1.

Theorem 2. Let the function

$$f(x) = \tilde{f}(x) + \sum_{\lambda > a_k} P_{m,k}(x_1)$$

$$\exp\left[i\sqrt{\lambda - a_k}\, x_1\right] \varphi_k(x^*),$$

where $\tilde{f}(x) \in H_{0,\beta}$, and $P_{m,k}(x_1)$ are polynomials of degree m.

For problem (1), (2) to have a solution

$$u(x) = \tilde{u}(x) + \sum_{\lambda > \alpha_k} Q_{m+1,k}(x_1) \exp[i\sqrt{\lambda - \alpha_k}\, x_1]\varphi_k(x^*),$$

where $\tilde{u}(x) \in H_{2,\beta}$, and $Q_{m+1,k}(x_1)$ are polynomials, it is necessary and sufficient that relation (6) be satisfied. Moreover, if the coefficients of the polynomials $P_{m,k}$ at x_1^l do not exceed in modulus $c_k \mu_k^l / l!$, where $\mu_k = \sqrt{\lambda - \alpha_k}$, then the coefficients of the polynomials $Q_{m+1,k}$ at x_1^l do not exceed in modulus $c_k \mu_k^{l-2} / l!$.

The proof of this theorem is analogous to the proof of Theorem 1. With its aid one can study perturbations of the eigenvalues and eigenfunctions of the operator $\mathcal{L}$.

Let $\mathcal{L}_1$ be the differential operator

$$\mathcal{L}_1 u \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a'_{ij}(x) \frac{\partial u}{\partial x_j} \right) + b'(x) u,$$

where the coefficients $a'_{ij}(x) \in C_1(\overline{D})$ and $b'(x) \in C(\overline{D})$ are real and are different from zero only in the domain D_0.

Theorem 3. Suppose that there exist r linearly independent solutions of the equation

$$\mathcal{L}u + \lambda_0 u = 0, \tag{8}$$

vanishing on S and belonging to $W_2^2(D)$, and that $\lambda_0 \neq \alpha_k$. Then, for sufficiently small ε, there exist r linearly independent functions $u_j(x, \varepsilon) \in W_2^2(D)$ which satisfy the equation

$$(\mathcal{L} + \varepsilon \mathcal{L}_1) u_j(x, \varepsilon) + \lambda_j(\varepsilon) u_j(x, \varepsilon) = 0, \qquad j = 1, 2, \dots, r, \tag{9}$$

where $u_j(x, \varepsilon)$ and $\lambda_j(\varepsilon)$ are analytic functions of the parameter ε, and $\lambda_j(0) = \lambda_0$.

We note that if $u(x) \in W_2^2(D)$ and satisfies equation (8), then $u(x) \in H_{2,\beta}$ for some $\beta > 0$. Expanding $\lambda_j(\varepsilon)$ and $u_j(x, \varepsilon)$ in series in ε, one can obtain equations for the coefficients of the expansion and indicate a method for constructing them. Having constructed majorant series, one can verify that the power series for $\lambda_j(\varepsilon)$ and $u_j(x,\varepsilon)$ converge. The function $u_j(x,\varepsilon)$ is a solution of equation (9), and

$$u_j(x,\varepsilon) = \tilde{u}_j(x,\varepsilon) + \sum_{\alpha_k<\lambda} v_{j,k}(x_1,\varepsilon)\exp[i\sqrt{\lambda-\alpha_k}\,x_1]\varphi_k(x^*),$$

where $\tilde{u}_j(x,\varepsilon) \in H_{2,\beta}$, and $v_{j,k}(x_1,\varepsilon)$ are entire functions of the variable x_1. Estimates of the order and type of these functions lead to the relation $v_{j,k}(x_1) \equiv 0$, which completes the proof of the theorem.

Thus, the eigenvalues of the operator are stable with respect to small finite perturbations of the operator if $\lambda \neq \alpha_k$ for every k.

If the eigenvalue λ_0 is located in the interval $\sigma_k = (\alpha_{k-1},\alpha_k)$, then the eigenvalue $\lambda_j(\varepsilon)$ is an analytic function in the interval $(-\varepsilon_0,\varepsilon_0)$ under the condition that, for these values, the operator $\mathcal{L}+\varepsilon\mathcal{L}_1$ is elliptic and $\lambda_j(\varepsilon)$ does not leave the interval σ_k. The only points at which the point spectrum is transformed into the continuous spectrum are the points α_k. If

$$\lambda_j(\varepsilon) \xrightarrow[\varepsilon\to\varepsilon_1]{} \alpha_k,$$

then one can choose a subsequence of eigenfunctions $u_j(x,\varepsilon_m)$ which, in every finite domain, converge in W_2^2, as $\varepsilon_m \to \varepsilon_1$, to a solution $u(x) \neq 0$ of the equation $\mathcal{L}u + \alpha_k u = 0$. The limiting eigenfunction, generally speaking, does not belong to $\mathcal{L}_2(D)$, but is bounded.

The theorems formulated above are obviously generalized also to the case in which the domain D outside a sufficiently large ball consists of several semi-infinite cylinders.

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Note: Figure translations are in progress. See original paper for figures.