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Abstract

Full Text

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ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF CERTAIN BOUNDARY-VALUE PROBLEMS FOR DEGENERATE ELLIPTIC EQUATIONS OF SECOND ORDER

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In the paper [1], M. Schechter proposed a sufficiently general method for proving the existence and uniqueness of the solution of the Dirichlet problem for degenerate elliptic equations of second order in the case when the boundary conditions on the "boundary of degeneration" are retained. In the present paper this method is generalized to the case when the boundary conditions on the boundary of degeneration are removed, and also to certain other boundary-value problems. As a special case, this yields the existence and uniqueness of the solution for the equations and boundary-value problems considered in [2-5].

1. Let D be a connected bounded domain in E_n, lying in the half-space $x_n \geq 0$. The boundary of the domain is $\dot{D} = \dot{D}' \cup \dot{D}''$, where $\dot{D}' = \dot{D} \cap (x_n > 0)$, $\dot{D}'' = \dot{D} \cap (x_n = 0)$; $\dot{D}'$ belongs to the class $C^{2+\alpha}$. In the domain D consider the differential equation of second order

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_j} + au = f, \tag{1}$$

where L is an operator elliptic at every point of the domain $D \cup \dot{D}'$. We shall assume that the coefficients of L belong to the class C^α in any domain $\overline{D}_\lambda = \overline{D} \cap (x_n \geq \lambda)$ for $\lambda > 0$.

On the boundary $\dot{D}'$ an operator is given,

$$\mathcal{R}u = -\mathcal{A}\frac{\partial u}{\partial \gamma} + \mathcal{B}u,$$

where the functions $\mathcal{A} \geq 0$, $\mathcal{B} > 0$ are defined and belong to the class $C^{1+\alpha}$ on $\dot{D}'$, and $\partial u/\partial\gamma$ is the derivative of u in some direction γ, forming an acute angle with the inward normal n to $\dot{D}'$.

Domains in which the operator L is uniformly elliptic will be denoted by G. In particular, we shall consider domains G_m (m a positive integer) with the following properties: the boundary $\dot{G}_m$ of the domain G_m belongs to the class $C^{2+\alpha}$, $\overline{D}_{1/m} \subseteq \overline{G}_m \subseteq \overline{G}_{m+1} \subset \overline{D}$, and $\dot{G}_m$ do not intersect the hyperplane

$x_n = 0$. We shall denote $\dot{G}'_m = \dot{G}_m \cap \dot{D}'$, $\dot{G}''_m = \dot{G}_m \setminus \dot{G}'_m$. We shall assume that the operator $\mathcal{R}$ and the boundary $\dot{D}'$ are such that there exists a continuous extension R_γ of this operator and of the field of directions γ in the domain D, possessing the following properties:

a) $R_\gamma = -A\,\partial/\partial\gamma + B$, where $A \geq 0$ and $B > 0$ for $x_n > 0$ belong to the class $C^{1+\alpha}$;

b) on the boundary $\dot{G}_m$, $\cos(\gamma, n) > 0$ for sufficiently large $m \geq m_0 > 0$;

c) for every $m \geq m_0$ in the domain G_m the boundary-value problem is solvable:

$$Lu = f \text{ in } G_m; \quad R_\gamma u = \Phi \text{ on } \dot{G}_m$$

for arbitrary $f \in C^\alpha(G_m)$ and $\Phi \in C^{1+\alpha}(\dot{G}_m)$ and

for the solution the Schauder estimate is valid

$$\|u\|_{2+\alpha}^{D_\lambda} \leqslant c\{\|f\|_\alpha^{G_m} + \|\Phi\|_{1+\alpha}^{\dot{G}_m} + \|u\|_0^{G_m}\},$$

where $\|u\|_{1+\alpha}^M$ is the norm of u in the space $C^{1+\alpha}(M)$, and $\overline{D}_\lambda \subset \overline{G}_m$ $(\lambda > 1/m)$. The constant c does not depend on u, f, and Φ.

Without loss of generality, we shall henceforth assume that the domain D lies in the strip $0 \leq x_n \leq 1$.

A function $\psi(x_n)$, twice continuously differentiable in $D \setminus \dot{D}'$, positive in D, will be called a **majorant** of equation (1) in D if, for any domain $G \subset D$, from the fact that for a solution of equation (1) on the boundary of G the conditions $|R_\gamma u| \leq R_\gamma \psi$ on $\dot{G}^{(1)}$ and $|u| \leq \psi$ on $\dot{G}^{(2)}$ $(\dot{G} = \dot{G}^{(1)} \cup \dot{G}^{(2)})$ are fulfilled, it follows that $|u| \leq \psi$ in G. If $f \equiv 0$, then the corresponding majorant will be called homogeneous; otherwise, nonhomogeneous.

Let $p(t)$ and $q(t) \geq 0$ be functions continuous on the semi-interval $(0, 1]$, and let $h(t, \mu, \nu)$ be the general solution of the equation $h'' + ph' + q = 0$, where μ, ν are arbitrary constants. The following is obvious.

Lemma 1. *In order that $h(t, \mu, \nu)$ be nonnegative on $[0, 1]$ for some μ, ν, it is necessary and sufficient that one of the following two pairs of conditions be fulfilled:*

$$\int_0^1 e^{P(r)} q(r)\,dr < \infty, \qquad \int_0^1 e^{P(s)}\,ds = \infty; \tag{2}$$

$$\int_0^1 e^{P(s)} \int_s^1 e^{-P(r)} q(r)\,dr\,ds < \infty, \qquad \int_0^1 e^{P(s)}\,ds < \infty, \tag{3}$$

where

$$P(s) = \int_s^1 p(r)\,dr.$$

Corollary. *If condition (2) is fulfilled, the function*

$$h_1(t) = \int_t^1 e^{P(s)} \int_0^s e^{-P(r)} q(r)\,dr\,ds + \nu_1$$

for any $\nu_1 > 0$ is a positive and monotonically decreasing function on $(0,1)$.

If condition (3) is fulfilled, the function

$$h_2(t) = \int_0^t e^{P(s)} \left[\int_s^1 e^{-P(r)} q(r)\,dr + \mu_2\right] ds$$

for any $\mu_2 > 0$ is positive and monotonically increasing on $(0,1)$.

If the operator R_γ on the boundary $\dot{G}^{(1)}$ of the domain $G \subset D$ is such that $A \geq 0$, $B > 0$, and $\cos(\gamma, n) > 0$ on $\dot{G}^{(1)}$, then the following is valid.

Lemma 2. *Let $|Lu| \leq \lambda q(x_n)$ $(\lambda \geq 0)$, $a_{nn} \equiv 1$, $a \leq 0$, $a_n \geq p(x_n)$ $[a_n \leq p(x_n)]$ in G, and suppose condition (2) [(3)] is fulfilled. Then from*

$$|R_\gamma u| \leq \lambda R_\gamma h_1 \quad [\,|R_\gamma u| \leq \lambda R_\gamma h_2\,] \quad \text{on } \dot{G}^{(1)},$$

$$|u| \leq \lambda h_1 \quad [\,|u| \leq \lambda h_2\,] \quad \text{on } \dot{G}^{(2)} = \dot{G} \setminus \dot{G}^{(1)}$$

it follows that

$$|u| \leq \lambda h_1 \quad [\,|u| \leq \lambda h_2\,] \quad \text{in } \overline{G}.$$

Corollary. *It follows from Lemma 2 that the function $\psi_1(x_n) = h_1(x_n)$ $[\psi_2(x_n) = h_2(x_n)]$ is a nonhomogeneous majorant of the equation $Lu = f$*

in D, if $|f| \leq q(x_n)$ and the remaining conditions of Lemma 2 are fulfilled. The function $\Psi_1(x_n) \equiv \lambda h_1(x_n)$ $[\Psi_2(x_n) = \lambda h_2(x_n)]$ is a homogeneous majorant of the equation $Lu = 0$ in D for any $\lambda > 0$, $q \equiv 0$, if the conditions of Lemma 2 are fulfilled in D.

We shall call the problem of finding a solution of equation (1) satisfying the condition $\mathcal{R}u = 0$ on $\dot{D}'$ the problem K_0.

Lemma 3 (existence). Suppose that in D there exists a continuous extension R_γ of the boundary operator $\mathcal{R}$, possessing properties a), b), c), and suppose that there exists an inhomogeneous majorant $\psi(x_n)$ of the equation $Lu = f$ in D such that $R_\gamma \psi \geq 0$ in D. Then there exists a solution u of problem K_0 in D, satisfying the estimate $|u| \leq \psi(x_n)$ in D.

Lemma 4 (uniqueness). The solution of problem K_0 is unique in the class of functions satisfying the condition $|u| \leq \psi(x_n)$ in D, if in D there exists a homogeneous majorant $\Psi(x_n)$ such that $R_\gamma \Psi \geq 0$ in D and $\psi(x_n)\{\Psi(x_n)\}^{-1} \to 0$ as $x_n \to 0$.

Choosing ν_1 $[\mu_2]$ sufficiently large and taking into account that $B > 0$ for $x_n > 0$, we can ensure that for $x_n \geq \varepsilon > 0$, $R_\gamma \psi_1 > 0$ $[R_\gamma \psi_2 > 0]$. If, moreover, we require that

$$\frac{A}{B} \leq \frac{1}{2} e^{-P(x_n)} \int_{x_n}^{1} e^{P(s)}\, ds \quad \text{for } 0 < x_n < \varepsilon \text{ and } \cos(\gamma, x_n) < 0, \tag{4}$$

then $R_\gamma \psi_1 > 0$ on the boundary of any domain G_m.

If instead we require that

$$\frac{A}{B} \leq \frac{1}{2} e^{-P(x_n)} \int_{0}^{x_n} e^{P(s)}\, ds \quad \text{for } 0 < x_n < \varepsilon \text{ and } \cos(\gamma, x_n) > 0, \tag{5}$$

then $R_\gamma \psi_2 > 0$ on the boundary of any domain G_m. The functions $\Psi_1(x_n)$ $[\Psi_2(x_n)]$ possess analogous properties.

With the aid of Lemmas 2, 3 and 4 one obtains

Theorem 1. Suppose $|f| \leq q(x_n)$; $a_{nn} \equiv 1$; $a \leq 0$; $a_n \geq p(x_n)$ $[a_n \leq p(x_n)]$, and condition (2) [(3)] is fulfilled. Suppose that in D there exists a continuous extension R_γ of the boundary operator R, possessing properties a), b) and c), and also satisfying condition (4) [(5)]. Then there exists a solution of problem K_0 in D, satisfying the estimate

$$|u| \leq \psi_1(x_n) \quad [\,|u| \leq \psi_2(x_n)\,] \quad \text{in } D. \tag{6}$$

This solution is unique in the class of functions satisfying estimate (6).

Corollary. Since $\psi_2(x_n) \to 0$ as $x_n \to 0$, when conditions (3) and (5) are fulfilled it follows from estimate (6) that the solution $u = 0$ on D''.

2. Consider boundary-value problems for equation (1) with nonhomogeneous boundary conditions. By the problem K_1 $[K_2]$ we shall mean the problem of finding a solution of equation (1) satisfying the condition $\mathcal{R}u = \varphi_1$ on $\dot{D}'$ $[\mathcal{R}u = \varphi_1$ on $\dot{D}'$, $u = \varphi_2$ on $\dot{D}'']$. The problem K_1 in essence does not differ from the problem K_0, if there exists an extension in D, $\Phi_1 \in C^{1+\alpha}(D)$, of the function φ_1, satisfying the estimate

$$|\Phi_1| \leq c_1 \left\{ \int_{x_n}^{1} e^{P(s)} \int_0^s e^{-P(r)} q(r)\, dr\, ds + \nu \right\}, \tag{7}$$

where c_1, ν are positive constants. Condition (7) is fulfilled, for example, if $\varphi_1 \in C^{1+\alpha}(\dot{D}')$ and is bounded in the closure of $\dot{D}'$.

When considering problem K_2, suppose that $\varphi_2 \in C^{2+\alpha}(\dot{D}'')$. Then, putting $\Phi_2(x_1, x_2, \dots, x_n) \equiv \varphi_2(x_n)$, by means of the substitution of the unknown function $u = w + \Phi_2$ we can reduce problem K_2 to an analogous problem, but with the condition $w = 0$ on $\dot{D}''$ and with the condition $\mathcal{R}w = \varphi_1 - \mathcal{R}\Phi_2$ on $\dot{D}'$. If there exists an extension to D, $\Phi \in C^{1+\alpha}(D)$, of the function $\varphi = \varphi_1 - \mathcal{R}\Phi_2$, satisfying the estimate

$$|\Phi| \leq c_2 \left\{ \int_0^{x_n} e^{P(s)} \int_s^1 e^{-P(r)} q(r)\, dr\, ds + \mu \int_0^{x_n} e^{P(s)}\, ds \right\}, \tag{8}$$

where c_2, μ are positive constants, then the construction of a solution of problem K_2 will be analogous to the construction of a solution of problem K_0.

Thus we arrive at the following theorem:

Theorem 2. Suppose that the conditions of Theorem 1 are fulfilled and that there exists an extension to D, $\Phi_1 \in C^{1+\alpha}(D)$ $[\Phi \in C^{1+\alpha}(D)]$, of the function $\varphi_1[\varphi = \varphi_1 - \mathcal{R}\Phi_2]$ given on $\dot{D}'$, satisfying estimate (7) [(8)]. Suppose, moreover, that in the case of problem K_2 the estimates $|a_{ik}|, |a_i|, |a| \leq q(x_n)$ hold for $i, k \neq n$. Then in D there exists a solution of problem $K_1[K_2]$, satisfying the estimate

$$|u| \leq \psi_1 \quad [|u - \varphi_2| \leq \psi_2], \quad \text{in } D.$$

This solution is unique in the class of functions satisfying estimate (9).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.