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Abstract

Full Text

MATHEMATICS

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ON THE REPRESENTATION OF AN ARBITRARY ANALYTIC FUNCTION BY SPECIAL SERIES

(Presented by Academician I. M. Vinogradov, 10 V 1965)

A. F. Leont'ev, in the work (1), considered the question of representing arbitrary analytic functions in a convex domain by Dirichlet series with complex exponents. In the present note the question is considered of representing arbitrary analytic functions in a ρ-convex domain by more general series.

1. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \neq 0, \tag{1}$$

be an entire function of finite order ρ and normal type σ, and suppose that for it the following condition is fulfilled: there exists the limit

$$\lim_{n\to\infty} \left(n^{1/\rho} \sqrt[n]{|a_n|}\right) = (\rho\sigma e)^{1/\rho}. \tag{2}$$

Let, further,

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \tag{3}$$

be an entire function of order ρ and type σ'. Put

$$\gamma(t) = \sum_{n=0}^{\infty} \frac{b_n}{a_n t^{n+1}}. \tag{4}$$

By virtue of condition (2), we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n/a_n|} = (\sigma'/\sigma)^{1/\rho}.$$

Consequently, the radius of convergence of series (4) is equal to $(\sigma'/\sigma)^{1/\rho}$. It is easy to verify that the following integral representation of the function $g(z)$ holds:

$$g(z) = \frac{1}{2\pi i} \int_L \gamma(t) f(zt)\, dt, \tag{5}$$

where L is a closed contour enclosing all singularities of the function $\gamma(t)$. We shall call the function $\gamma(t)$ the function associated with the function $g(z)$ with respect to the function $f(z)$.

2. Let $L(z)$ be an entire function of order $\rho > 1/2$ and normal type σ, and let $h(\varphi) > 0$ be its indicator of growth. Suppose that the following condition is fulfilled:

There is a system of circles $|z| = \rho_k,\ k = 1, 2, \dots,\ \rho_k \uparrow \infty$, such that

$$\frac{\ln |L(re^{i\varphi})|}{r^\rho} > h(\varphi) - \varepsilon, \qquad r = \rho_k, \quad k > K(\varepsilon). \tag{6}$$

By $\gamma(t)$ we denote the function associated with the function $L(z)$ by means of the function $E_\rho(z)$, where

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + n\rho^{-1})}. \tag{7}$$

Consider in the z-plane the family of curves

$$\mathrm{Re}(ze^{-i\varphi})^\rho = r^\rho \cos \rho(\theta - \varphi) = \tau, \qquad |\theta - \varphi| \le \pi/2\rho, \tag{8}$$

where $z = re^{i\theta}$, and the parameters φ and τ vary respectively in the intervals $[0, 2\pi]$ and $[0, +\infty)$. In (8) we consider that branch of the function $(ze^{-i\varphi})^\rho$ which assumes positive values on the ray $\arg z = \varphi$. Each curve of the family (8) divides the z-plane into two infinite domains. The closure of each of these domains will be called an elementary ρ-convex domain.

A point set M will be called ρ-convex (this definition is given in (2)) if it can be represented as the intersection of a finite or infinite number of elementary ρ-convex domains. The intersection of all ρ-convex domains containing the set M will be called the smallest ρ-convex hull of the set M.

Let $\overline{D}$ be the smallest ρ-convex hull of the set of all singular points of the function $\gamma(t)$, and let D be the set of interior points of the set $\overline{D}$.

Let $F(z)$ be an arbitrary function holomorphic on the closed set $\overline{D}$. Put

$$\omega_F(\mu) = \frac{1}{2\pi i} \int_C \frac{F(t)}{t}\, dt \int_0^\infty \frac{L(x/t) - L(\mu)}{x/t - \mu}\, d\tau(x), \tag{9}$$

where $\tau(x) = -e^{-x^\rho}$, C is a closed contour such that $F(z)$ is regular on C and inside C, and $\overline{D}$ lies inside C.

Theorem 1. On every closed subset F of the set D the estimate

$$\left| F(z) - \frac{1}{2\pi i} \int_{|\mu|=\rho_k} \frac{\omega_F(\mu) E_\rho(\mu z)}{L(\mu)}\, d\mu \right| \le e^{-\delta \rho_k^\rho}, \qquad z \in F, \quad k > K_0(F), \tag{10}$$

holds, where $\delta > 0$ depends on the set F and on the function $F(z)$.

Remark. In any domain G containing all singularities of the function $\gamma(t)$, the system $\{z^k E_\rho^{(k)}(\lambda_\nu z),\ k = 0, 1, 2, \dots,\ \rho_\nu - 1\}_{\nu=1}^{\infty}$, where λ_ν is a zero of the function $L(\lambda)$ of order ρ_ν, is incomplete. It follows from this that in such a domain G the estimate (10) cannot hold for every function analytic in the domain G.

In particular, when in each annulus $\rho_{k-1} < |z| < \rho_k$, $k = 1, 2, \dots$, $\rho_0 = 0$, there is one zero of the function $L(z)$, it follows from the estimate (10) that the function $F(z)$ expands into a series.

Let us note the main stages of the proof of Theorem 1. Put

$$\Phi_k(z, \lambda) = L(\lambda) \frac{1}{2\pi i} \int_{|\mu|=\rho_k} \frac{E_\rho(z\mu)}{(\mu - \lambda) L(\mu)}\, d\mu, \tag{11}$$

where λ lies inside the circle $|\mu| = \rho_k$. The function $\Phi_k(z, \lambda)$, as a function of λ, is analytically continuable to the whole plane. Consequently, it is an entire function. It can be proved that the function $\Phi_k(z, \lambda)$, as a function of λ, for $z \in \overline{D}$, has order ρ and indicator $h(\varphi)$. Denote by $\gamma_k(z, t)$ the function associated with the function $\Phi_k(z, \lambda)$, as a function of λ, with respect to the function $E_\rho(z)$. As a function of t, $\gamma_k(z, t)$ is regular everywhere outside $\overline{D}$. We assert that the equality

$$F(z) - \frac{1}{2\pi i} \int_{|\mu|=\rho_k} \frac{\omega_F(\mu) E_\rho(\mu z)}{L(\mu)}\, d\mu = \frac{1}{2\pi i} \int_C \gamma_k(z, t) F(t)\, dt, \qquad z \in \overline{D}. \tag{12}$$

holds.

Let us first verify this equality for functions $F(t)$ of the form $E_\rho(\lambda z)$, where λ is a fixed number, with $|\lambda| < \rho_k$. In doing this we shall use the fact that, for $f(z) = E_\rho(z)$, the representation (5) admits the inverse

$$\gamma(t) = \frac{1}{t} \int_0^\infty g\left(\frac{x}{t}\right) d\tau(x), \tag{13}$$

where $\tau(x) = -e^{-x^\rho}$. From (9) and (13) it follows that, for $F(z) = E_\rho(\lambda z)$, we have

$$\omega_F(\mu) = \frac{L(\lambda) - L(\mu)}{\lambda - \mu},$$

and therefore the left-hand side of relation (12) is equal to

$$E_\rho(\lambda z) - \frac{1}{2\pi i}\int_{|\mu|=\rho_k} \frac{E_\rho(\mu z)}{\mu - \lambda}\, d\mu + \frac{L(\lambda)}{2\pi i}\int_{|\mu|=\rho_k} \frac{E_\rho(\mu z)}{(\mu - \lambda)L(\mu)}\, d\mu = \Phi_k(z, \lambda).$$

The right-hand side of relation (12) is also equal to $\Phi_k(z, \lambda)$. Since there exists a system of functions $\{E_\rho(\lambda_i z)\}_{i=1}^\infty$, complete in the whole plane, where $|\lambda_i| < \rho_k$, and since both sides of relation (12) depend continuously on the function $F(z)$, relation (12) is also valid for an arbitrary function $F(z)$ analytic on C and inside C. Thus relation (12) has been established. Using the integral representation for $\gamma_k(z, t)$, analogous to representation (13), one can show that the function $\gamma_k(z, t)$, for $z \in F$ and $t \in C$, has the estimate

$$|\gamma_k(z, t)| \le e^{-\delta \rho_k^\rho}, \quad k > K_0(F), \tag{14}$$

where $\delta > 0$ depends on the set F and on the function $F(z)$. Hence, from equality (12), Theorem 1 follows.

3. Let $L(\lambda)$ be an entire function of order ρ and type σ_1, possessing property (6), and let $h(\varphi) > 0$ $(0 \le \varphi \le 2\pi)$ be its growth indicator. Further, let $f(z)$ be an entire function possessing properties (1) and (2). By $\gamma(t)$ we denote the function associated with the function $L(\lambda)$, with respect to the function $f(z)$. The function $\gamma(t)$ is regular outside the circle

$$|t| \le (\sigma_1/\sigma)^{1/\rho}. \tag{15}$$

Let $F(z)$ be an analytic function in the closed circle (15). Put

$$\omega_F(\mu) = \frac{1}{2\pi i}\int_C F(t)\gamma(t, \mu)\, dt, \tag{16}$$

where C is a circle of radius a, $a > (\sigma_1/\sigma)^{1/\rho}$; $\gamma(t, \mu)$ is the function associated with the function $(L(\lambda) - L(\mu))/(\lambda - \mu)$, as a function of λ, with respect to the function $f(z)$.

Theorem 2. In any disk of radius β

$$\beta < \left[\min_{\varphi} h(\varphi)/\sigma\right]^{1/\rho}$$

the estimate

$$\left|F(z) - \frac{1}{2\pi i}\int \frac{\omega_F(\mu) f(\mu z)}{L(\mu)}\, d\mu\right| \le e_k^{-\delta_0 \rho}, \qquad |z| \le \beta, \qquad k > K_0(\beta), \qquad (17)$$

holds, where $\delta > 0$ depends on β and on the function $F(z)$.

Theorem 2 is proved essentially by a scheme analogous to the proof of Theorem 1.

In conclusion I express my gratitude to Prof. A. F. Leont' ev for posing the problem and for a number of valuable suggestions.

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Note: Figure translations are in progress. See original paper for figures.

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