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Abstract

Full Text

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ON NONLINEAR EQUATIONS OF THE FIRST KIND

The purpose of the present article is to develop methods for solving equations of the first kind

$$A[z] = u \quad (z \in Z, u \in U),$$

where $A[z]$ is a continuous operator from Z into U , satisfying the uniqueness condition: $A[z_1] \neq A[z_2]$, if $z_1 \neq z_2$, and Z and U are metric spaces satisfying the conditions formulated below. These methods are a development of the ideas set forth in ⁽¹⁻³⁾.

1. We shall assume that in Z there exists a set \bar{Z} , which admits a new metrization $\rho_1(\bar{z}_1, \bar{z}_2)$, majorant with respect to the metrization $\rho_0(z_1, z_2)$ of the space Z :

$$\rho_0(\bar{z}_1, \bar{z}_2) \leq \rho_1(\bar{z}_1, \bar{z}_2),$$

and, moreover, the sphere $\bar{S}_C(\bar{z}_0) = \{\bar{z} : \rho_1(\bar{z}, \bar{z}_0) \leq C\}$ is compact in Z in the sense of the metric ρ_0 . We shall say that the space \bar{Z}_1 (the set \bar{Z} in the metric ρ_1) is s -compactly embedded in Z .

Let the space \bar{Z}_1 , s -compactly embedded in Z , be a convex closed set of some Hilbert space, so that $\rho_1(\bar{z}_1, \bar{z}_2) = (\bar{z}_1 - \bar{z}_2, \bar{z}_1 - \bar{z}_2)_1 = \|\bar{z}_1 - \bar{z}_2\|_1$, and let the concept of a segment be defined

$$[\bar{z}_1, \bar{z}_2] = \{\bar{z} = \alpha\bar{z}_1 + \beta\bar{z}_2 \in \bar{Z}, \alpha > 0, \beta > 0, \alpha + \beta = 1\}.$$

We shall assume that the diameter of the segment $[\bar{z}_1, \bar{z}_2]$ in the norm ρ_0 tends to zero if $\rho_0(\bar{z}_1, \bar{z}_2) \rightarrow 0$, which is always fulfilled if the space Z is normed.

The spaces $Z = C(a, b)$ and $\bar{Z} = W_2^1(a, b)$, or

$$\bar{Z} = \{\bar{z}(s) : \bar{z} \in W_2^1, q(s) \leq \bar{z}(s) \leq p(s)\},$$

where $q(s)$ and $p(s)$ are prescribed bounded functions, satisfy these conditions.

Consider the smoothing functional

$$M^\alpha[\bar{z}, \tilde{u}] = \rho_2^2(A[\bar{z}], \tilde{u}) + \alpha\Omega[\bar{z}] \quad (\bar{z} \in \bar{Z}),$$

where \tilde{u} is an arbitrary element of U ; $\rho_2(u_1, u_2)$ is the metric of U , $\Omega[\bar{z}] = \|\bar{z}\|^2$, and $\alpha > 0$ is a numerical parameter.

Theorem 1. *If \bar{Z} is a closed convex set of a Hilbert space, then for every \tilde{u} and $\alpha > 0$ there exists an element $z^\alpha \in \bar{Z}$ realizing the minimum $M^\alpha[\bar{z}, \tilde{u}]$.*

Let

$$M_0 = \inf M^\alpha[\bar{z}, \tilde{u}], \quad \bar{z} \in \bar{Z},$$

and let \bar{z}_n be a minimizing sequence

$$M_n = M^\alpha[\bar{z}_n, \tilde{u}] \rightarrow M_0 \quad (\text{as } n \rightarrow \infty).$$

Without loss of generality one may assume that $M_{n-1} \leq M_n$. Then we have

$$\Omega(\bar{z}_n) \leq C_1^2 \quad \left(C_1^2 = \frac{1}{\alpha} M_1 \right),$$

i.e. $\bar{z}_n \in \bar{S}_{C_1}$ for every n . Without changing the notation, we shall assume that the sequence \bar{z}_n converges to z^α in the sense of the metric ρ_0 . It is not difficult to see

show that $z^\alpha \in \bar{Z}$. Indeed, for any ε we have $\|\bar{z}_n - \bar{z}_{n+p}\|_1 \leq \varepsilon$, if $n \geq n(\varepsilon)$ and $p > 0$. Suppose that this is not so, i.e., that there exists an ε_0 such that

$$\|\bar{z}_n - \bar{z}_{n+p_n}\|_1 \geq \varepsilon_0$$

for arbitrarily large indices n . Put $\xi_n = \bar{z}_n - \bar{z}_{n+p_n}$. Obviously,

$$\rho_0(\bar{z}_n, \bar{z}_{n+p_n}) \rightarrow 0 \quad (n \rightarrow \infty).$$

Consider the element $\xi_n = \frac{1}{2}(z_n + z_{n+p_n}) = z_n - \frac{1}{2}\xi_n = \bar{z}_{n+p_n} + \frac{1}{2}\xi_n$ and

$$M^\alpha[\xi_n, u] = \rho_2^2(A[\bar{z}_n - \frac{1}{2}\xi_n], \tilde{u}) + \alpha(\|\bar{z}_n\|_1^2 - (\bar{z}_n, \xi_n)_1 + \frac{1}{4}\|\xi_n\|_1^2) \geq M_0.$$

Since $\rho_0(\xi_n, z^\alpha) \rightarrow 0$ and $\rho_0(\xi_n, \bar{z}_n) \rightarrow 0$, we have

$$\rho_2^2(A[\xi_n], \tilde{u}) = \rho_2^2(A[\bar{z}_n], \tilde{u}) + \Delta_n^{(1)} \quad (\Delta_n^{(1)} \rightarrow 0 \text{ as } n \rightarrow \infty),$$

whence it follows that

$$\alpha(-(\dot{z}_n, \xi_n)_1 + \frac{1}{4}\|\xi_n\|_1^2) \geq M_0 - M_n - |\Delta_n^{(1)}| = -\Delta'_n,$$

$$(\Delta'_n > 0 \text{ and } \Delta'_n \rightarrow 0, n \rightarrow \infty).$$

Similarly, representing ξ_n as $\bar{z}_{n+p_n} + \frac{1}{2}\xi_n$, we obtain

$$\alpha((\bar{z}_{n+p_n}, \xi_n)_1 + \frac{1}{4}\|\xi_n\|_1^2) \geq -\Delta''_n \quad (\Delta''_n > 0 \text{ and } \Delta''_n \rightarrow 0, n \rightarrow \infty),$$

and also

$$\alpha(-(\bar{z}_n - \bar{z}_{n+p_n}, \xi_n)_1 + \frac{1}{2}\|\xi_n\|_1^2) = -\frac{1}{2}\alpha\|\xi_n\|_1^2 \geq -(\Delta'_n + \Delta''_n)$$

or

$$\|\xi_n\|_1 = \|\bar{z}_n - \bar{z}_{n+p_n}\|_1 \leq \varepsilon_0/2 \quad \text{for } n \geq n(\varepsilon),$$

which contradicts the assumption. By the completeness of the space \bar{Z} , the element $z^\alpha \in \bar{Z}$.

Theorem 2. Let $\bar{u}^{(0)} = A[\bar{z}^{(0)}]$, where $\bar{z}^{(0)} \in \bar{Z}$ (\bar{Z} satisfies the conditions of Theorem 1), and let $\rho_2(\bar{u}, \bar{u}^{(0)}) \leq \delta$. Then for any $\varepsilon > 0$ and any function $\alpha_0(\delta)$: $\alpha_0(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and $\alpha_0(\delta) \geq q_0\delta^2$, there exists a $\delta_0(\varepsilon, \alpha_0, q_0)$ such that any element z^α realizing the minimum of $M^\alpha[\bar{z}, \bar{u}]$ in \bar{Z} satisfies the condition

$$\rho_0(z^\alpha, \bar{z}^{(0)}) \leq \varepsilon, \quad \text{if } q_0\delta^2 \leq \alpha \leq \alpha_0(\delta).$$

Indeed,

$$M^\alpha[z^\alpha, \tilde{u}] \leq M^\alpha[\bar{z}^{(0)}, \tilde{u}] \leq \delta^2 + \alpha\|\bar{z}^{(0)}\|_1^2,$$

whence:

$$1^\circ. \quad \|z^\alpha\|^2 \leq \frac{1}{\alpha}(\delta^2 + \alpha\|\bar{z}^{(0)}\|_1^2) \leq \frac{1}{q_0} + \|\bar{z}^{(0)}\|_1^2 = C^2 \quad (\text{i.e. } z^\alpha \in \bar{S}_C).$$

$$2^\circ. \quad \rho_2(A[z^\alpha], A[\bar{z}^{(0)}]) \leq \rho_2(A[z^\alpha], \tilde{u}) + \delta \leq \delta + \sqrt{\delta^2 + \alpha_0(\delta)\|\bar{z}^{(0)}\|_1^2} = \eta(\delta).$$

Since all z^α and $\bar{z}^{(0)}$ belong to the compact set K_C —the closure of \bar{S}_C in Z —there exists an $\bar{\eta}(\varepsilon)$ such that if $\rho_2(u^\alpha, \bar{u}^{(0)}) \leq \bar{\eta}(\varepsilon)$, then $\rho_0(z^\alpha, \bar{z}^{(0)}) \leq \varepsilon$. Choosing δ_0 so that $\eta(\delta) \leq \bar{\eta}(\varepsilon)$ for $\delta \leq \delta_0$, we shall have

$$\rho_0(z^\alpha, \bar{z}^{(0)}) \leq \varepsilon \quad \text{for } \delta \leq \delta_0, \text{ if } q\delta^2 \leq \alpha \leq \alpha_0(\delta).$$

- II. 2. If one does not make the assumption of completeness of the space \bar{Z} , then the assertion that $z^\alpha \in \bar{Z}$ is false. However, if \hat{z}^α denotes some element for which

$$M^\alpha[\hat{z}^\alpha, \tilde{u}] \leq M_0 + \alpha C \quad (C = \text{const}),$$

then the following holds.

Theorem 3. Let \bar{Z} be s -compactly embedded in Z , $\bar{u}^{(0)} = A[\bar{z}^{(0)}]$, where $\bar{z}^{(0)} \in \bar{Z}$. Then, for any ε and any function $\alpha_0(\delta) : \alpha_0(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, and $\alpha_0(\delta) \geq q_0\delta^2$, there exists a $\delta_0(\varepsilon, \|\bar{z}\|, \alpha_0, q_0, C)$ such that any element \hat{z}^α for which

$$M^\alpha[\hat{z}^\alpha, \tilde{u}] \leq M_0 + \alpha C \quad (\rho_2(\tilde{u}, \bar{u}^{(0)}) \leq \delta),$$

satisfies the condition

$$\rho_0(\hat{z}^\alpha, \bar{z}^{(0)}) \leq \varepsilon, \quad \text{if } q_0\delta^2 \leq \alpha \leq \alpha_0(\delta).$$

The proof of this theorem is close in idea to the proof of Theorem 2.

- II. 3. In the present subsection we shall give a condition under which z^α is determined uniquely.

Theorem 4. If Z is normed, \bar{Z} and U are Hilbert spaces, $A[\bar{z}^{(0)}] = \bar{u}^{(0)}$, and the operator $A(z) = u$ has a first differential $A[z, \xi]$ satisfying the conditions:

- 1) $\|A_1[z_1, \xi] - A_1[z_2, \xi]\|_2 \leq A_2[\bar{z}^{(0)}]\|z_1 - z_2\|_0\|\xi\|_0$, if $\|z_1 - \bar{z}^{(0)}\|_0 \leq \varepsilon$, $\|z_2 - \bar{z}^{(0)}\|_0 \leq \varepsilon$;
- 2) $A_1[\bar{z}^{(0)}, \xi] \equiv 0$ only for $\xi \equiv 0$,

and if $q_0\delta^2 \leq \alpha \leq \alpha_0(\delta)$, then there exists a $\delta_1(\|\bar{z}^{(0)}\|_1, q_0, \alpha_0)$ such that for every function \tilde{u} such that $\|\tilde{u} - \bar{u}^{(0)}\|_2 \leq \delta_1$, the function z^α is determined uniquely.

Let z^α be some element realizing the minimum

$$M^\alpha[\bar{z}, \tilde{u}] = \|A[\bar{z}] - \tilde{u}\|_2^2 + \alpha\|\bar{z}\|_1^2.$$

In this case

$$\delta M^\alpha[z^\alpha, \xi, \tilde{u}] = (A[z^\alpha] - \tilde{u}; A_1[z^\alpha, \xi])_2 + \alpha(z^\alpha, \xi) = 0.$$

If there exist two elements z_1^α and z_2^α for which δM^α , then, taking their difference $v = z_1^\alpha - z_2^\alpha$, we obtain

$$R = \delta M^\alpha[z_1^\alpha, \xi, \tilde{u}] - \delta M^\alpha[z_2^\alpha, \xi, \tilde{u}] = (A[z_1^\alpha] - A[z_2^\alpha]; A_1[z_1^\alpha, \xi])_2 + \\ + (A[z_2^\alpha] - \tilde{u}; A_1[z_1^\alpha, \xi] - A_1[z_2^\alpha, \xi])_2 + \alpha(v, \xi)_1 = 0.$$

We shall use the fact that

1°. $A[z_1^\alpha] - A[z_2^\alpha] = A_1[\bar{z}^{(0)}, v] + \hat{u}$, $\|\hat{u}\|_2 \leq \eta(\delta)\|v\|_0$ ($\eta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$), since z_1^α and z_2^α , by Theorem 2, for sufficiently small δ , lie in an ε -neighborhood of $\bar{z}^{(0)}$.

2°. $\|A_1[z_1^\alpha, \xi] - A_1[z_2^\alpha, \xi]\|_2 \leq A_2\|v\|_0\|\xi\|_0$.

3°. $A_1[z_1^\alpha, \xi] = A_1[\bar{z}^{(0)}, \xi] + \hat{u}$, $\|\hat{u}\|_2 \leq \eta(\delta)\|v\|_0$.

4°. $\|A[z_2^\alpha] - \tilde{u}\|_2 \leq \eta(\delta)$.

Putting $\xi = v$, we shall have

$$R = \|A_1[\bar{z}^{(0)}, v]\|_2^2 + r + \alpha\|v\|_1^2 = 0, \quad |r| \leq \eta(\delta)\|v\|_0^2.$$

Let us verify that

$$\|A_1[\bar{z}^{(0)}, v]\|_2^2 \geq s_0^2\|v\|_0^2,$$

from which it will follow that $v \equiv 0$.

Indeed, in the proof of Theorem 2 it was established that $\|z^\alpha\|_1 \leq C$; thus $\|v\|_1 \leq 2C$. Consider

$$s_0 = \inf \|A_1[\bar{z}^{(0)}, \xi]\|_2, \quad \xi \in \bar{Z}\{\xi : \|\xi\|_0 = 1, \|\xi\|_1 \leq 2C\}.$$

We shall show that $s_0 > 0$. Let ζ_n be a minimizing sequence: $\|A_1[\bar{z}^{(0)}, \zeta_n]\|_2 = s_n \rightarrow s_0$ (as $n \rightarrow \infty$). In view of the fact that \bar{Z} is compact in Z , we may assume that ζ_n converges in the metric $\|\cdot\|_0$ to some element ζ_0 , and, by the continuity of $A_1[\bar{z}^{(0)}, \zeta]$ with respect to ζ , in the metric $\|\cdot\|_0$ we shall have $\|A_1[\bar{z}^{(0)}, \zeta_0]\|_2 = s_0$, whence we conclude that $s_0 \neq 0$, since $\|\zeta_0\|_0 = 1$.

The proof given is based on the fact that the functional $M^\alpha[z, \tilde{u}]$ has only a single stationary point for sufficiently small α . With the help of this proposition, the method of steepest descent is justified for sufficiently small α .

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Note: Figure translations are in progress. See original paper for figures.

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