\(C^{1,\alpha}\)-Isometric Immersions of Riemannian Spaces

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Abstract

Full Text

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$C^{1,\alpha}$-Isometric Immersions of Riemannian Spaces

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Let V^n be an n-dimensional Riemannian space with an analytic metric, i.e., an n-dimensional analytic manifold on which an invariant positive definite quadratic form $g_{ij}dx^i dx^j$ of coordinate differentials with analytic coefficients g_{ij} is given.

A mapping of V^n into an m-dimensional Euclidean space E^m $(m > n)$ will be called a $C^{l,\alpha}$-isometric immersion of the space V^n if it is given by functions $u^k = u^k(x^1, \dots, x^n)$ (x^i are analytic coordinates in V^n; u^k are Cartesian rectangular coordinates in E^m), subject to the following conditions:

1) the u^k have continuous partial derivatives up to order l inclusive $(l \geq 1)$, satisfying a Hölder condition with exponent α $(0 \leq \alpha \leq 1)$ in a neighborhood of each point;

2) the matrix
$$\begin{pmatrix} \partial u^1/\partial x^1 & \dots & \partial u^m/\partial x^1 \\ \cdot & \cdot & \cdot \\ \partial u^1/\partial x^n & \dots & \partial u^m/\partial x^n \end{pmatrix}$$
has rank n everywhere;

3)
$$\sum_{k=1}^{m} \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} = g_{ij}$$
(the condition of isometry of the immersion)$*$.

It is not difficult to observe that this definition depends neither on the choice of the coordinates x^i in V^n, nor on the choice of the coordinates u^k in E^m.

The surface defined by the parametric equations $u^k = u^k(x^1, \dots, x^n)$ will be called a $C^{l,\alpha}$-realization of the space V^n in the space E^m. If the functions $u^k(x^1, \dots, x^n)$ are analytic, the corresponding immersions and realizations are called analytic.

We shall say that a $C^{l,\alpha}$-isometric immersion has regularity $C^{l,\alpha}$ if at no point is it a $C^{l,\alpha'}$-isometric immersion for which $\alpha' > \alpha$. The set of all immersions having regularity $C^{l,\alpha}$ will be called the class of immersions $C^{l,\alpha}$. The set of all analytic immersions will be called the class A. A class of immersions will be called universal for spaces of dimension n if every space V^n admits a local immersion of the given class into $E^{n+1}$$**$. For spaces of dimension 1, obviously, every class of immersions is universal. As is known from differential geometry, the class A is universal for spaces of dimension 2, but is not so for spaces of higher dimensions. Do there exist, for spaces of dimension $n \geq 2$, universal classes distinct from A, and what are they? This question has a twofold meaning.

* If in this definition, while preserving the conditions of existence and continuity of $\partial u^k/\partial x^i$, as well as 2) and 3), one replaces condition 1) by the requirement that in some neighborhood of the point M there exist continuous derivatives up to order l, satisfying the Hölder condition with exponent α only at the point M itself, then one obtains the definition of an immersion $C^{l,\alpha}$-isometric at the point M.

** That is, every point of V^n has a neighborhood admitting such an immersion.

If $n = 2$, then the question is whether one can prescribe in advance to smooth surfaces a certain degree of regularity completely independently of the properties of their intrinsic metric, and, if so, within what limits. Such a possibility is necessarily connected with the violation of any relation between the intrinsic and extrinsic geometry of a surface which ensures local convexity, and thereby also the analyticity of a surface with an analytic metric of positive curvature. Among such relations are, in particular, the equality of the integral of the modulus of the Gaussian curvature over the area and the variation of the spherical image of the surface (see ([1])).

For $n > 2$ the question is also one of whether, by weakening the regularity of the immersion, one can ensure the local immersibility of any space V^n.

The first of the results available here is, in essence, contained (although not stated explicitly) in the works of Nash ([2]) and Kuiper ([3]) on $C^{1,0}$-isometric immersions of Riemannian spaces. It is expressed in the following theorem:

Theorem 1. *The class of immersions $C^{1,0}$ is universal for spaces of any dimension.*

Moreover, from the theorem that a $C^{1,\alpha}$-realization of a space V^2 of positive curvature in E^3 is analytic for $\alpha > 2/3$ (see ([4])), it follows that

Theorem 2. *For any $l > 1$ and arbitrary α, and also for $l = 1$ and $\alpha > 2/3$, the class of immersions $C^{l,\alpha}$ is not universal for spaces V^2.*

This theorem, as is not difficult to understand, gives a certain lower estimate for the "degree of waviness" of surfaces corresponding to universal immersions. The following theorem gives an estimate of the opposite character.

Theorem 3. *For any $\alpha < 1/(n^2 + n + 1)$, the class of immersions $C^{1,\alpha}$ is universal for spaces V^n.*

For $n = 2$ the restriction on α has the form $\alpha < 1/7$. Apparently, in the case $n = 2$, Theorem 3 can be strengthened by replacing the inequality $\alpha < 1/7$ with the inequality $\alpha < 1/5$.

Whether Theorem 2 is true for dimensions $n > 2$ is unknown. The result known from differential geometry on the existence of spaces V^n, $n > 2$, which do not admit local immersions of class $C^{l,\alpha}$, $l \geqslant 3$, can, by simple arguments, be extended to immersions of class $C^{2,\alpha}$.

The study of smooth surfaces on which Levi-Civita parallel transport of a vector is defined (see ([5])), and the analysis of the Nash–Kuiper construction of $C^{1,0}$-isometric immersions, make the following hypothesis plausible:

For spaces of any dimension, the classes of immersions $C^{1,\alpha}$ with $\alpha \leqslant 1/2$ are universal; there are no other universal classes (except the class A for spaces V^2).

As already noted, immersions of a space V^2 of positive Gaussian curvature belonging to a nonanalytic class are such that the corresponding smooth surfaces are not locally convex at any point. At the same time, for the indicated spaces, a neighborhood of each point homeomorphic to a disk admits an analytic immersion in the form of a convex surface analytically continuable beyond the boundary of this neighborhood. There exists a continuous bending of this convex surface into a surface homeomorphic to it, corresponding to an immersion of class $C^{1,\alpha}$, where α is any number less than $1/7$, i.e., a continuous bending with violation of local convexity at every point. More precisely, the following holds:

Theorem 4. Let U be a domain of the space V^2 of positive Gaussian curvature, homeomorphic to a disk. Suppose, further, that the functions $u^i(x^1, x^2)$, $i = 1, 2, 3$, define an analytic immersion of the domain $\bar{U}_1 \supset \bar{U}$ in E^3 in the form of a convex surface. For any α, $0 \le \alpha < 1/7$, there exists a family of functions $u^i_t(x^1, x^2)$, $i = 1, 2, 3$, depending on a parameter t, $0 \le t \le 1$, defined on the domain U and possessing the following properties:

1) for every $t > 0$ the functions $u^i_t(x^1, x^2)$ define an isometric immersion of U of class C^1 into E^3 in the form of a surface homeomorphic to a disk, and the constant in the Hölder condition does not depend on x^1, x^2 or t;
2) the functions $u^i_t(x^1, x^2)$, together with the derivatives $\partial u^i_t/\partial x^1$, $\partial u^i_t/\partial x^2$, are uniformly continuous in t $(0 \le t \le 1)$ in the domain U;
3) for $t = 0$ the functions $u^i_t(x^1, x^2)$ coincide with $u^i(x^1, x^2)$.

Apparently, Theorem 4 can be strengthened by replacing the inequality $\alpha < 1/7$ by the inequality $\alpha < 1/5$. The theorem on a continuous bending of a closed convex analytic surface, analogous to Theorem 4, is valid for values $\alpha < 1/13$. Apparently, this inequality can be replaced by the inequality $\alpha < 1/9$.

Finally, let us note that the violation of the usual relations between the intrinsic and extrinsic geometry of smooth surfaces corresponding to immersions of class $C^{1,\alpha}$, $0 \le \alpha < 1/7$, is not exhausted by the violation of the local convexity of surfaces of positive Gaussian curvature. One can point out also such an "effect" as the existence of homeomorphic isometric immersions of the Euclidean plane into E^3, belonging to any of the classes $C^{1,\alpha}$, $0 \le \alpha < 1/7$, such that the corresponding surface is not ruled. Moreover, there exists a continuous bending of the plane into such a non-ruled surface. The exact formulation of this result is entirely analogous to Theorem 4. Here, apparently, it is also possible to replace the inequality $\alpha < 1/7$ by the inequality $\alpha < 1/5$.

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Note: Figure translations are in progress. See original paper for figures.