A PRIORI ESTIMATES AND SOLVABILITY OF GENERAL BOUNDARY VALUE PROBLEMS FOR GENERAL ELLIPTIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

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Abstract

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A PRIORI ESTIMATES AND SOLVABILITY OF GENERAL BOUNDARY VALUE PROBLEMS FOR GENERAL ELLIPTIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

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1. Here we set forth the result of a study of general boundary value problems for systems elliptic in the sense of Douglis–Nirenberg with discontinuous coefficients. These results were obtained by means of a method developed in recent years in a number of works [1–6], and are a generalization to systems of the investigations of M. Schechter [12], Ya. A. Roitberg and Z. G. Sheftel' * [1,3].

2. **Notation, statement of the problem.** Let G be a bounded domain of the space E_n with boundary Γ, divided by a surface γ into two subdomains ** G_1 and G_2, with γ and Γ having no common points. Let, for definiteness, γ be the boundary of G_1.

We shall consider the spaces (see [1–3]) $W_p^l(G) = W_p^l(G_1) + W_p^l(G_2)$ ($l \geq 0$ an integer, $p > 1$) and the spaces $C^{l+\alpha}(G) = C^{l+\alpha}(G_1) + C^{l+\alpha}(G_2)$ ($0 < \alpha < 1$) of piecewise smooth functions with norm (see [4])

$$|v|_{l+\alpha} = |v|_{l+\alpha}^{G} = |v|_{l+\alpha}^{G_1} + |v|_{l+\alpha}^{G_2}.$$

By $C^{l+\alpha}(\gamma)$ and $C^{l+\alpha}(\Gamma)$ we shall mean the spaces of Hölder functions on γ and Γ (see [4]), and by $v^{(m)}(x)$ the value of the function $v(x)$ for $x \in G_m$ ($m = 1, 2$).

We define the operators

$$\mathcal{L}(x, \partial/\partial x) = \begin{cases} \mathcal{L}^{(1)}(x, \partial/\partial x), & x \in G_1, \\ \mathcal{L}^{(2)}(x, \partial/\partial x), & x \in G_2; \end{cases} \tag{1}$$

$$B^{(1)}(x, \partial/\partial x) \text{ and } B^{(2)}(x, \partial/\partial x) \text{ on } \gamma; \tag{2}$$

$$\mathcal{H}(x, \partial/\partial x) \text{ on } \Gamma. \tag{3}$$

$\mathcal{L}^{(m)}(x, \partial/\partial x)$, $B^m(x, \partial/\partial x)$ $(m = 1, 2)$, $\mathcal{H}(x, \partial/\partial x)$ are matrices of respective sizes $N \times N$, $2r \times N$, $r \times N$ (the numbers r and N will be determined below), whose elements $l_{\sim m)}(x, \partial/\partial x)$, $b_{qj}^{(m)}(x, \partial/\partial x)$ $(m = 1, 2)$, $h_{kj}(x, \partial/\partial x)$ are linear differential operators of orders $s_i + t_j$, $\sigma_q + t_j$, $m_k + t_j$. The s_i, t_j, σ_q, m_k are integers, with $s_i \leq 0$, $t_j \geq 0$ $(i, j = 1, \dots, N)$. Denote by $\mathcal{L}_0(x, \partial/\partial x)$, $B_0^{(m)}(x, \partial/\partial x)$ $(m = 1, 2)$, $\mathcal{H}_0(x, \partial/\partial x)$ the operators obtained from the operators (1), (2), (3) by discarding in $l_{ij}(x, \partial/\partial x)$, $b_{qj}^{(m)}(x, \partial/\partial x)$, $h_{kj}(x, \partial/\partial x)$ all terms containing differentiations of orders lower than $s_i + t_j$, $\sigma_q + t_j$, $m_k + t_j$; and by $\mathcal{L}_0(x, \xi)$, $B_0^{(m)}(x, \xi)$, $\mathcal{H}_0(x, \xi)$ the matrices obtained from the corresponding operators by replacing $\partial/\partial x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ by $\xi = (\xi_1, \dots, \xi_{n-1}, \tau) = (\xi', \tau)$. The operator

* After the present work had been completed, it became known to the author that analogous results had been obtained by Z. G. Sheftel' .

** All results are also valid in the case of a partition of G into a finite number of domains.

$\mathcal{L}(x, \partial/\partial x)$ is called properly elliptic in $\overline{G}$ (see (8)) if, for any $x \in \overline{G}_m$ and any real vector $(\xi', \tau) \neq 0$,

$$L^{(m)}(x, \xi', \tau) = \det(\mathcal{L}_0^{(m)}(x, \xi', \tau)) \neq 0, \tag{4}$$

and the roots τ of the polynomials (4), for any real $\xi' \neq 0$, are equally distributed in the upper and lower half-planes. From proper ellipticity it follows that the order of $L(x, \xi', \tau)$ is an even number, $\sum(s_i + t_i) = 2r$.

Below we shall consider only properly elliptic operators.

The coefficients entering into $l_{ij}, b_{qj}^{(m)}, h_{kj}$ belong respectively to the spaces

$$C^{l-s_i+\alpha}(G),\ C^{l-\sigma_q+\alpha}(\gamma),\ C^{l-m_k+\alpha}(\Gamma), \quad \text{where } l > l_0 = \max_{q,k}(|\sigma_q|, |m_k|).$$

Consider the problem of finding a solution of the system

$$\mathcal{L}(x, \partial/\partial x)u(x) = f(x), \quad x \in G - \gamma, \tag{5}$$

satisfying on γ the conjugation conditions

$$B^{(1)}(x, \partial/\partial x)u^{(1)}(x) - B^{(2)}(x, \partial/\partial x)u^{(2)}(x) = \varphi(x), \quad x \in \gamma, \tag{6}$$

and on Γ the boundary conditions

$$\mathcal{H}[x, \partial/\partial x)u(x) = g(x), \quad x \in \Gamma. \tag{7}$$

For vector-functions $u(x) = (u_1, \dots, u_N)$, $f(x) = (f_1, \dots, f_N)$, introduce the spaces $\mathfrak{U}^{(l)}$ and $\mathfrak{F}^{(l)}$ in G as direct products of the spaces

$$\mathfrak{U}^{(l)} = \prod_{j=1}^{N} W_p^{l+t_j}(G), \quad \mathfrak{F}^{(l)} = \prod_{i=1}^{N} W_p^{l-s_i}(G) \quad (l \geq l_0 + 1) \tag{8}$$

with norms

$$\|u, \mathfrak{U}^{(l)}\| = \sum_{j=1}^{N} \|u_j\|_{l+t_j}, \quad \|f, \mathfrak{F}^{(l)}\| = \sum_{i=1}^{N} \|f_i\|_{l-s_i}. \tag{9}$$

By $\Phi^{(l-1/p)}$ and $\mathcal{G}^{(l-1/p)}$ we denote the direct products of the spaces of L. N. Slobodetskii (see (7)) on γ and Γ

$$\Phi^{(l-1/p)} = \prod_{q=1}^{2r} W_p^{l-\sigma_q-1/p}(\gamma), \quad \mathcal{G}^{(l-1/p)} = \prod_{k=1}^{r} W_p^{l-m_k-1/p}(\Gamma) \tag{10}$$

with norms

$$\|\varphi, \Phi^{(l-1/p)}\| = \sum_{q=1}^{2p} \|\varphi_q\|_{l-\sigma_q-1/p}, \quad \|g, \mathcal{G}^{(l-1/p)}\| = \sum_{k=1}^{r} \|g_k\|_{l-m_k-1/p}. \tag{11}$$

By $\mathfrak{U}_c^{(l+\alpha)}$, $\mathfrak{F}_c^{(l+\alpha)}$, $\Phi_c^{(l+\alpha)}$, $\mathcal{G}_c^{(l+\alpha)}$ we shall understand the analogous spaces composed of $C^{l+t_j+\alpha}(G)$, $C^{l-s_i+\alpha}(G)$, $C^{l-\sigma_q+\alpha}(\gamma)$, $C^{l-m_k+\alpha}(\Gamma)$, with the corresponding norms.

Denote by

$$B(x, \partial/\partial x)u(x) = B^{(1)}(x, \partial/\partial x)u^{(1)} - B^{(2)}(x, \partial/\partial x)u^{(2)}$$

on γ, and associate with problem (5)–(7) the operator $\mathfrak{A} = (\mathcal{L}, B, \mathcal{H})$, acting boundedly from

$$\mathfrak{U}_c^{(l+\alpha)} \to H_c^{(l+\alpha)} = \mathfrak{F}_c^{(l+\alpha)} \times \Phi_c^{(l+\alpha)} \times \mathcal{G}_c^{(l+\alpha)} \quad \text{for } l \geq l_0,$$

$$\mathfrak{U}^{(l)} \to H^{(l)} = \mathfrak{F}^{(l)} \times \Phi^{(l-1/p)} \times \mathcal{G}^{(l-1/p)} \quad \text{for } l \geq l_0 + 1.$$

3. Condition of joint covering of operators.
Let $t = \max t_j$, let $l \geq l_0$ be an integer, and let the surface $\gamma(\Gamma)$ belong to the class

$C^{l++\alpha}$. For any point $P \in \gamma(\Gamma)$, write all operators in such a local coordinate system $\tilde{x}$ that the axis $\tilde{x}_n$ is directed along the inward normal to $\gamma(\Gamma)$, and the remaining axes lie in the tangent plane. Simplifying the notation, we again denote the local coordinates by x. Denote by $\widehat{\mathcal{L}_0}(x, \xi', \tau)$ the matrix adjoint to $\mathcal{L}_0(x, \xi', \tau)$, and

$$C_m(x, \xi', \tau) = B_0^{(m)}(x, \xi', \tau) \cdot \widehat{\mathcal{L}}_0^{(m)}(x, \xi', \tau) \quad (m = 1, 2). \tag{12}$$

It follows from proper ellipticity that

$$L^m(x, \xi', \tau) = L_+^{(m)}(x, \xi', \tau) \cdot L_-^{(m)}(x, \xi', \tau) \quad (m = 1, 2) \tag{13}$$

(the τ-roots of $L_+^{(m)}$ (L_-^m) have $\operatorname{Im}\tau > 0$ ($\operatorname{Im}\tau < 0$) for any real $\xi' \neq 0$).

Let $C_1'(x, \xi', \tau)$ and $C_2'(x, \xi', \tau)$ be the matrices composed of the remainders from division (with respect to τ) of $C_1(x, \xi', \tau)$ and $C_2(x, \xi', \tau)$ respectively by $L_+^{(1)}$ and $L_-^{(2)}$. Following the terminology of ([1−3]), we introduce the notion of joint covering of operators.

Definition. The operators (2) and (3) **jointly cover** the properly elliptic operator $\mathcal{L}(x, \partial/\partial x)$ if:
1) for any point $x \in \gamma$ and any real vector $\xi' \neq 0$, the rows of the matrix

$$C' = (C_1'(x, \xi', \tau),\, C_2'(x, \xi', \tau))$$

are linearly independent with respect to τ;
2) the operator $\mathcal{H}(x, \partial/\partial x)$ on Γ satisfies the Lopatinskii condition ([5,6]).

4. Estimates of Schauder type. Let $l \geq l_0$.

Theorem 1. Let the vector-function $u(x) \in \mathfrak{U}^{(l+\alpha)}$ be a solution of problem (5)–(7), $\mathcal{L}u \in \mathfrak{F}_c^{(l+\alpha)}$, $Bu \in \Phi_c^{(l+\alpha)}$, $\mathcal{H}u \in \mathcal{G}^{(l+\alpha)}$.

The joint covering condition is necessary and sufficient for $u(x) \in \mathfrak{U}_c^{(l+\alpha)}$,

$$|u, \mathfrak{U}_c^{(l+\alpha)}| \leq A_1(|f, \mathfrak{F}_c^{(l+\alpha)}| + |\varphi, \Phi_c^{(l+\alpha)}| + |g, \mathcal{G}^{(l+\alpha)}| + |u|_0), \tag{14}$$

where

$$|u|_0 = \sum_{j=1}^{N} |u_j|_0,$$

and A_1 does not depend on f, φ, g.

The following assertion holds (see ([11])):

Theorem 2. 1) The range $R(\mathfrak{A})$ of the operator $\mathfrak{A}$ is closed in $H_c^{(l+\alpha)}$:

$$R(\mathfrak{A}) = \overline{R(\mathfrak{A})}.$$

2) The homogeneous problem (5)–(7) has in $H_c^{(l+\alpha)}$ a finite number of linearly independent solutions.

5. A priori estimates in L_p. In this section we shall assume that all the conditions imposed above are fulfilled, with $l \geq l_1 = l_0 + 1$.

Theorem 3. Let $u(x) \in \mathfrak{U}^{(l)}$ be a solution of problem (5)–(7),

$$\mathcal{L}u \in \mathfrak{F}^{(l)}, \quad Bu \in \Phi^{(l-1/p)}, \quad \mathcal{H}u \in \mathcal{G}^{(l-1/p)}.$$

The joint covering condition is necessary and sufficient for $u(x) \in \mathfrak{U}^{(l)}$ and

$$\|u, \mathfrak{U}^{(l)}\| \leq A_2(\|f, \mathfrak{F}^{(l)}\| + \|\varphi, \Phi^{(l-1/p)}\| + \|g, \mathcal{G}^{(l-1/p)}\| + \|u\|_0), \tag{15}$$

where

$$\|u\|_0 = \sum_{j=1}^{N} \|u_j\|_0,$$

and A_2 does not depend on f, φ, g.

We shall call the operator $\mathfrak{A}$ of problem (5)–(7) **elliptic** if $\mathcal{L}(x, \partial/\partial x)$ is properly elliptic and the operators (2), (3) jointly cover it.

The following theorem is valid (see ([9,11])):

Theorem 4. The following assertions are equivalent:

1) the operator $\mathfrak{A}$ is elliptic;
2) if $u \in \mathfrak{U}^{(l)}$, $\mathcal{L}u \in \mathfrak{F}^{(l)}$, $Bu \in \Phi^{(l-1/p)}$, $\mathcal{H}u \in \mathcal{G}^{(l-1/p)}$, then $u(x) \in \mathfrak{U}^{(l)}$ and the estimate (15) holds;
3) the operator $\mathfrak{A}$ is a Φ-operator in the sense of (10).

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Note: Figure translations are in progress. See original paper for figures.