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Abstract

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STATISTICAL MECHANICS OF A ONE-DIMENSIONAL SYSTEM WITH COULOMB INTERACTION

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Introduction. Recently a number of works have appeared devoted to one-dimensional statistical models ($^{1-3}$), which is explained by the possibility of a rather complete study of their thermodynamic properties and by possible applications (^{4}). In the case of interaction of nearest neighbors only, the statistical integral is calculated exactly for an arbitrary interaction potential (^{5}). In a number of American papers, systems with a potential of the form $\Phi(x) = \alpha x$ (one-dimensional "Coulomb") ($^{1-3}$), arising in the three-dimensional case in the interaction of infinite parallel charged planes, are studied. Stability is ensured by attaching a reservoir of particles with the opposite charge. To solve the problem, the apparatus of probability theory is used; however, attempts at an exact solution encounter considerable mathematical difficulties. In the present work a system with a potential of the same type is investigated, but stability is ensured at the expense of "polarization" of the interacting elements. An asymptotically exact expression is obtained for the configurational integral of the system.

The system is in a state of thermodynamic equilibrium and consists of identical elements $1, \dots, N$, located at points $x_1, \dots, x_N$ on the axis Ox. In addition, the elements with numbers 0 and $N+1$ are fixed immovably ($x_{N+1} \equiv L$) and realize the walls. It is assumed that

$$0 \le x_1 \le x_2 \le \dots \le x_N \le L.$$

The interaction is effected through the potential

$$\Phi(|x_i - x_j|) = \alpha\gamma^{|i-j|-1}|x_i - x_j|.$$

In the case $\gamma = 1$ the force of interaction does not depend on the mutual distance between the i-th and j-th elements; $0 < \gamma < 1$ means a weakening of the interaction by a factor of $1/\gamma$, if between two elements there is still some other one; finally, $\gamma = 0$ means that only nearest neighbors interact.

The configurational integral of the system may be written in the form

$$\bar{Q}_N = N! \int_0^L dx_N \dots \int_0^{x_2} \exp(-\beta U_{N+1})\, dx_1, \tag{1}$$

where $\beta = 1/kT$,

$$U_{N+1} = \sum_{(i<j)} \alpha \gamma^{|i-j|-1} |x_i - x_j|.$$

The last expression can be represented in the form

$$U_{N+1} = \alpha \sum_k C_k (x_{k+1} - x_k), \tag{2}$$

where

$$C_k = \begin{cases} (N+1-k)(1+k), & \gamma = 1, \\ (\gamma^{k+1} - 1)(\gamma^{N-k+1} - 1)/(\gamma - 1)^2, & 0 < \gamma < 1, \\ 1, & \gamma = 0. \end{cases}$$

Note that $C_k = C_{N-k}$.

1°. **Proof of the absence of an equation of state of the system for $\gamma = 1$.** Introduce the functions

$$f_k(x_{k+1} - x_k) = \exp[-\alpha\beta C_k (x_{k+1} - x_k)]$$

and consider

$$Q_N \equiv \overline{Q}_N / N! = \int_0^L dx_N \dots \int_0^{x_2} \prod_{(k)} f_k(x_{k+1} - x_k)\, dx.$$

We can write a sequence of functions obtained as the convolution of the preceding one with f_M:

$$F_M(x) = \int_0^x f_M(x) F_{M-1}(x - \xi)\, d\xi \quad (M = 2, 3, \dots); \qquad F_1 = f_1(x).$$

Then $\overline{Q}_N = F_{N+1}(L)$.

Denote by $\tilde{f}(s)$ the Laplace transform of $f(s)$,

$$\tilde{f}_k(s) = \int_0^\infty e^{-sx} f_k(x)\, dx \qquad (\mathrm{Re}\{s\} > 0).$$

Note that, by virtue of the property $C_k = C_{N-k}$, we have

$$\tilde{f}_k(s) = \tilde{f}_{N-k}(s) = [s + \alpha\beta(N+1-k)(1+k)]^{-1}.$$

Obviously,

$$Q_N(L) = \frac{1}{2\pi i} \oint e^{sL} \prod_{k=0}^{N/2} [s + \alpha\beta C_k]^{-2}\, ds.$$

For simplicity one may take N to be an even number. It will be clear from what follows that this assumption is inessential.

On the axis $\mathrm{Re}\{s\}$, the position of the j-th pole is determined by the equality

$$s_j = -\alpha\beta(N+1-j)(1+j),$$

and the residue at the j-th pole $2\pi i R_j$ has the form

$$2\pi i R_j = \frac{\rho_j}{(\alpha\beta)^N} \left[L - \frac{2}{\alpha\beta} \sum_{\substack{k=0\\(k\neq j)}}^{N/2} \frac{1}{(k-j)(N-k-j)} \right],$$

$$\rho_j = \exp[-\alpha\beta C_{jL}] \Big/ \prod_{\substack{k=0\\(k\neq j)}}^{N/2} (k-j)^2(N-k-j)^2.$$

Obviously,

$$Q_N(L) = \sum_{(j)} R_j. \tag{3}$$

We note the following properties of the expression for R_j:

a) $\left|\sum[(k-j)(N-k-j)]^{-1}\right| < \sum \dfrac{1}{k^2}$;

b) $\prod(k-j)^2(N-k-j)^2 = \left[\left(\dfrac{N}{2} - j\right)!\right]^4 \{(N-j)!\}^2\{j!\}^2$;

c)
$$\frac{\rho_j}{\rho_{j+1}} = \left[\frac{j+1}{(N/2-j)(N-j)}\right]^2 \exp[(N-2j-1)\alpha\beta L].$$

These properties ensure a rapid decrease of ρ_j with increasing j for $j < N/2$. For example,

$$\rho_0/\rho_1 = (4/N^4)\exp[(N-1)\alpha\beta L],$$

$$\rho_{N/2-1}/\rho_{N/2} = (N/(N+2))\exp(\alpha\beta L).$$

On the basis of a), b), c), as $L, N \to \infty$ one may conclude

$$\bar{Q}_N(L) \to \frac{e^{-\alpha\beta}(N+1)L}{N!^2(N/2!)^2}\,\frac{L}{(\alpha\beta)^N}$$

or, using Stirling' s formula and assuming $\lim\limits_{\substack{L\to\infty \\ N\to\infty}} (N/L) = l$, one can write

$$\bar{Q}_N(L) = \frac{2^{N-1}l}{\pi^2(\alpha\beta)^N}\,\frac{\exp[3N - \alpha\beta(N+1)L]}{(N)^{3N+1}}. \tag{4}$$

The equation of state is obtained with the aid of the known relation

$$P\beta = \frac{\partial}{\partial L}\ln \bar{Q}_N(L), \tag{5}$$

where P is the pressure in the system. With the aid of (4) it is not difficult to obtain

$$P\beta = -\left(2\alpha\beta N + \frac{3}{l}\ln N\right),$$

which means that, as $N \to \infty$, there is no equation of state for the system.

2°. **The equation of state for** $0 < \gamma < 1$**.** Using the method of item 1°, it would not be difficult to obtain

$$R_j = \frac{L(\gamma-1)^{2N}}{(\alpha\beta\gamma)^N}\exp(-\alpha\beta C_{jL}) \Big/ \prod_{\substack{k=0 \\ (k\neq j)}}^{N/2} (\gamma^j - \gamma^k)(\gamma^{N-(k+j)} - 1)^2,$$

and then apply formula (3). However, we shall use a simpler method. Suppose that $\gamma \neq a^{1/N}$. Choose some $\delta \ll 1$ and put $\gamma^{\chi+1} = \delta$. Denote

$$\chi = (\ln\delta/\ln\gamma) - 1 \qquad (\chi \ll N).$$

Then

$$C_k = C_{N-k} \to \frac{1-\delta}{(\gamma-1)^2} \to \frac{1}{(\gamma-1)^2} \qquad (0 < \gamma \le \delta)$$

for $k \gg \chi$ and $\delta \to 0$.

Repeating the reasoning of item 1°, it is easy to obtain:

$$Q_N(L) = \frac{1}{2\pi i} \oint e^{sL} T_\chi^2 (\tilde{f}(s))^{N-2\chi}\, ds, \tag{6}$$

where

$$T_\chi = \prod_{k=0}^{\chi} [s + \alpha\beta C_k]^{-1}, \qquad \tilde{f}(s) = \left[s + \frac{\alpha\beta}{(\gamma-1)^2} \right]^{-1}.$$

Denote

$$A(s) = e^{sL} T_\chi^2 \big(\tilde{f}(s)\big)^{N-2\chi} = e^{N\chi(s)},$$

where

$$\begin{gathered} \chi(s) = sl + \frac{2}{N} \ln T_\chi + \left(1 - \frac{2\chi}{N}\right) \ln \tilde{f}(s), \\ \chi(s) \to sl + \ln \tilde{f}(s) \qquad (N \to \infty). \end{gathered} \tag{7}$$

We use the saddle-point method. For the saddle point s_0 we have the condition

$$\chi' = l + \left[\frac{d}{ds} \ln \tilde{f}(s) \right]_{s=s_0} = 0. \tag{8}$$

Following the argument of (6), one can easily obtain $s_0 = p\beta$ and

$$\lim_{\substack{N\to\infty \\ L\to\infty}} Q_N(L, p, \beta) = e^{p\beta L} T_\chi^2(p\beta) \left[p\beta + \frac{\alpha\beta}{(\gamma-1)^2} \right]^{-N+2\chi},$$

and with the aid of (8) the equation of state

$$l\beta \left[p + \alpha(\gamma-1)^{-2} \right] = 1. \tag{9}$$

3. Interaction of neighboring planes. By the method set forth above, the expression for Q_N is obtained trivially:

$$Q_N = \frac{l^N}{\sqrt{2\pi N}} \exp\{(1-\alpha\beta l)N\}; \qquad \bar{Q}_N = (L)^N \exp(-\alpha\beta L). \tag{10}$$

Here we have used Stirling' s formula. Note that the result can just as simply be obtained directly from (1). With the aid of (5) and (10) we obtain the equation of state

$$l\beta(p+\alpha) = 1. \tag{11}$$

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Note: Figure translations are in progress. See original paper for figures.