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Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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A PARAMETRIC METHOD FOR SOLVING FRACTIONAL PROGRAMMING PROBLEMS ON ANALOG COMPUTERS

(Presented by Academician V. A. Trapeznikov on 18 XI 1964)

1°. Consider the nonlinear programming problem

$$\lambda^* = \min\{P(x),\ x \in R\} \tag{1}$$

with fractional objective function $P(x) = F_1(x)/F_2(x)$ and a nonempty, admissible, bounded and closed set R, defined by the system of constraints

$$\varphi_s(x) \le 0, \qquad s = 1, \dots, m, \tag{2}$$

$$\psi_j(x) = \sum_{k=1}^{n} a_{jk}x_k + b_j = 0, \qquad j = 1, \dots, q;\ q < n,$$

$$\alpha_i \le x_i \le \beta_i, \qquad i = 1, \dots, n,$$

where $x \in E_n$; $F_1(x), \varphi_s(x)$ are convex functions; $a_{jk}, b_j, \alpha_i, \beta_i$ are real numbers.

It is assumed that a segment of values of the parameter numbers $\Omega = [\lambda_1, \lambda_2]$ is known, among which lies the extremal value (1), $\lambda^* \in \Omega$. Further, it is assumed that $F_2(x) > 0$ for $x \in R$, and that the auxiliary function $\xi(x, \lambda) = -\lambda F_2(x)$, $\lambda \in \Omega$, is convex.

In addition, it is assumed that the functions $F_1(x), F_2(x), \varphi_s(x)$ are defined everywhere, continuously differentiable, and that their partial derivatives in every bounded closed domain satisfy a Lipschitz condition. The fractional programming problem is: find a vector $x = x^*$, $x^* \in R$, for which the objective function $P(x)$ attains the extremal value λ^*.

Remark 1. If $F_2(x) = 1$ (convex programming) or is linear, then there is no need to know the segment of numbers Ω. In this case the function $\xi(x, \lambda)$ is convex for any λ.

Remark 2. The problem formulated includes the known problem of fractional-linear programming ([1,2]) under additional constraints ($2'$).

2°. Introduce the function $V(x,\lambda)$ with parameter λ (λ is a real number)

$$V(x,\lambda) = \frac{1}{2}\{[F_1(x) - \lambda F_2(x)]^2 \operatorname{sg}[F_1(x) - \lambda F_2(x)] + \sum_{s=1}^{m} \varphi_s^2(x) \operatorname{sg} \varphi_s(x) + \sum_{j=1}^{q} \psi^2(x) + \sum_{i=1}^{n} [(x_i - \beta_i)^2 \operatorname{sg}(x_i - \beta_i) + (\alpha_i - x_i) \operatorname{sg}(\alpha_i - x_i)]\},$$

where $\operatorname{sg} r = 1$ for $r > 0$ and $\operatorname{sg} r = 0$ for $r \leq 0$. Let Ω' be the set of values of the function $P(x)$ on the set R, $\lambda^* \in \Omega'$ and $\Omega \cap \Omega' \neq 0$.

The function $V(x,\lambda)$ is everywhere convex and attains an absolute minimum on the compact convex set $Q(\lambda)$, moreover

$$V(x,\lambda) = 0 \quad \text{for } x \in Q(\lambda),\ Q(\lambda) \subseteq R,\ \lambda \in \Omega', \tag{3}$$

$$V(x,\lambda) > 0 \quad \text{in all other cases.}$$

From the convexity of $V(x,\lambda)$ and the boundedness of the set $Q(\lambda)$ it follows that

$$\lim_{\|x\| \to +\infty} V(x,\lambda) = +\infty, \tag{4}$$

$$\|\operatorname{grad} V(x,\lambda)\| > 0 \quad \text{when } V(x,\lambda) > 0. \tag{5}$$

If $\lambda = \lambda^*$, then the set $Q(\lambda^*)$ consists precisely of the points at which the extremal value $P(x) = \lambda^*$ is attained.

Using the properties (3) of the function $V(x,\lambda)$, the fractional-programming problem can be reduced to finding the minimum value of the parameter $\lambda \in \Omega$ and the corresponding vector x for which $V(x,\lambda) = 0$.

The procedure for finding $\lambda = \lambda^*$ and $x = x^*$ reduces to the following. For $\lambda = \lambda_1$ the function $V(x,\lambda_1)$ is minimized. If $\min_x V(x,\lambda_1) = 0$, then $\lambda_1 = \lambda^*$. If, however, $\min_x V(x,\lambda) > 0$, then the minimization is repeated for an increasing sequence of values of the parameter λ until such a λ_i is found for which $\min_x V(x,\lambda_i) = 0$ and $\min_x V(x,\lambda_i - \varepsilon) > 0$ for any arbitrarily small $\varepsilon > 0$. In this case $x = x^*$, $\lambda_i = \lambda^*$.

3°. To find the minimum of $V(x, \lambda)$ for fixed λ, we use the gradient system of differential equations

$$\tau_i \frac{dx_i}{dt} = -\frac{\partial V(x, \lambda)}{\partial x_i}, \qquad \tau_i > 0, \qquad i = 1, \ldots, n. \tag{6}$$

The solutions of this system $x(x_0, \lambda, t)$, independently of the initial conditions x_0, $x_0 \in Q(\lambda)$, for $t > 0$ and $t \to +\infty$ converge to the set $Q(\lambda)$, i.e., for any δ-neighborhood of the set $Q(\lambda)$ there exists a $T(\delta, x^0) > 0$ such that, for $t > T(\delta, x^0)$, $x(x_0, \lambda, t) \in \delta$ and $\|x(x_0, \lambda, t)\| < M$, $M > 0$.

Indeed, by virtue of (5) and (6),

$$\frac{dV(x, \lambda)}{dt} = \sum_{i=1}^{n} \frac{\partial V(x, \lambda)}{\partial x_i} \frac{dx_i}{dt} \leq -\frac{1}{\max_i\{\tau_i\}} \|\operatorname{grad} V(x, \lambda)\|^2 < 0 \quad \text{when } x \notin Q(\lambda),$$

$$dV(x, \lambda)/dt \equiv 0 \quad \text{when } x \in Q(\lambda).$$

It follows from (4) that every solution $x(x^0, \lambda, t)$ is bounded. The set $Q(\lambda)$ is an invariant set for the system (6), $dV(x(x_0, \lambda, t), \lambda)/dt \equiv 0$, if $x^0 \in Q(\lambda)$ and $x \in Q(\lambda)$ for $0 \leq t < +\infty$, and is an ω-limit set for its solutions, which proves assertion (3). In this case the function $V(x, \lambda)$ is an analogue of the Lyapunov function.

Taking (2) and (2′) into account, system (6) takes the form

$$-\tau_i \frac{dx_i}{dt} = \left\{ \left[\frac{\partial F_1(x)}{\partial x_i} - \lambda \frac{\partial F_2(x)}{\partial x_i} \right] [F_1(x) - \lambda F_2(x)] \operatorname{sg} [F_1(x) - \lambda F_2(x)] + \right.$$

$$+ \sum_{s=1}^{m} \frac{\partial \varphi_s(x)}{\partial x_i} \varphi_s(x) \operatorname{sg} \varphi_s(x) + \sum_{j=1}^{q} \frac{\partial \psi_j(x)}{\partial x_i} \psi_j(x) +$$

$$\left. + \sum_{i=1}^{n} [(x_i - \beta_i) \operatorname{sg}(x_i - \beta_i) - (\alpha_i - x_i) \operatorname{sg}(\alpha_i - x_i)] \right\}. \tag{7}$$

The system of differential equations (7) can be solved by known methods on an electronic analog computer (electronic model) [4]. With a sufficiently slow change of the parameter λ, the analog model will automatically track, with some error μ, the coordinates of the minimum of the function $W(x, \lambda)$, with $\mu \to 0$ as the rate of change of λ is decreased.

At the moment when the minimum of the function $V(x, \lambda)$ passes from a nonzero value to a zero value, the vector x and the parameter λ are recorded. The obtained values x, λ are the solution of the problem.

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Note: Figure translations are in progress. See original paper for figures.

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