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Abstract

Full Text

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On the Existence of Unbounded Analytic Constructive Functions

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1. I. D. Zaslavskii (1) constructed an example of a function* that is continuous on the segment $-\pi\triangle\pi$ and unbounded on this segment (Theorem 5.1). By a slight change in the construction given in the proof of Theorem 5.1 one can obtain infinite differentiability of the mentioned function on $-\pi\triangle\pi$. In connection with this the following question arises: do there exist constructive functions that are analytic inside the unit disk and continuous in the closed unit disk, and unbounded in this disk? (The concepts of a constructive function of a complex variable and of analyticity are specified in § 2.)

In this note a positive answer to this question is given.

2. Following the usual path, we introduce complex FR-numbers as pairs of FR-numbers. For this purpose denote by Ψ_4 the alphabet $\Psi_3 \cup \{\diamond\}$ (Ψ_3 is the alphabet of FR-numbers ((2), p. 77)).

A word C in the alphabet Ψ_4 will be called a **complex FR-number** if C is an FR-number or $C \doteq x_1 \diamond x_2$, where x_1 and x_2 are FR-numbers.

An algorithm φ in the alphabet Ψ_4^{ca} will be called a **constructive function of a complex variable** (c.f.c.v.) if, for any complex FR-numbers z_1 and z_2, from $!\varphi(z_1)$ and $z_1 = z_2$ it follows that $!\varphi(z_2)$ and $\varphi(z_1), \varphi(z_2)$ are equal complex FR-numbers.

In classical analysis the following two definitions of analyticity of a function of a complex variable turn out to be equivalent: a) a function is called analytic in a domain D if it is expandable into a power series in some neighborhood of every point of the domain D (such functions are sometimes called holomorphic in D; see (3), p. 54); b) a function is called analytic in a domain D if it is differentiable at every point of the domain D (such functions are sometimes called monogenic in D; see (3), p. 77).

In constructive analysis, even in the case of the simplest domains (for example, a disk), the question of the equivalence of these definitions apparently remains open. We shall use the term "analytic" for c.f.c.v.' s satisfying definition b); c.f.c.v.' s satisfying definition a) we shall call holomorphic.

Below the adjective "constructive" will often be omitted.

3. Let Φ be the sequence of systems of rational segments constructed according to Theorem 4.2 $^{(1)}$. From the proof of Theorem 4.2 the following properties of the sequence Φ are easily seen: 1) for any n the segments of the system Φ_n pairwise have no common points; 2) the sum of the lengths of all segments of the system Φ_n does not exceed $(2/5)^n$. Moreover, $\Phi_0 \doteq 0\triangle 1$ and $\Phi_{n+1} \subset \Phi_n$ for every n.

For $n \geq 1$ the system Φ_n has the form

$$\Phi_n \doteq r_{n,1}\triangle q_{n,1} * r_{n,2}\triangle q_{n,2} * \dots * r_{n,k_n}\triangle q_{n,k_n},$$

where

$$0 < r_{n,1} < q_{n,1} < \dots < r_{n,j} < q_{n,j} < \dots < r_{n,k_n} < q_{n,k_n} < 1,$$

and k_n is the number of segments of the system Φ_n. Denote by the system of segments

$$-\pi\triangle r_{n,1} * q_{n,1}\triangle r_{n,2} * \dots * q_{n,k_n-1}\triangle r_{n,k_n} * q_{n,k_n}\triangle\pi$$

through Ψ_n. Since $\Phi_{n+1} \subset \Phi_n$, for every $n \geq 1$ we have $\Psi_n \subset \Psi_{n+1}$. Moreover, the segments of the system Ψ_n have no common points pairwise.

We shall say that a function g is **Riemann integrable** (**R-integrable**) on the system Ψ_n if it is Riemann integrable on every segment of the system Ψ_n (Riemann integration is understood in accordance with (4)).

The FR-number equal to the sum of the Riemann integrals of the function g over all segments of the system Ψ_n will be called the **Riemann integral** (**R-integral**) of the function g over the system Ψ_n.

4. Consider the series $\sum_{k=1}^{\infty} a_k$, where a_k are FR-numbers. It is said that the series

$$\sum_{k=1}^{\infty} a_k$$

converges if one can construct a convergence regulator for this series.

It is said that the series $\sum_{k=1}^{\infty} a_k$ converges absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

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* The function constructed by I. D. Zaslavskii is already unbounded on $0\triangle 1$, but we shall not need this.

It is said that the series $\sum_{k=1}^{\infty} a_k$ converges to the FR-number y if this FR-number is the limit of the sequence $\sum_{k=1}^{n} a_k$ as $n \to \infty$. In this case the FR-number y is called the sum of the series $\sum_{k=1}^{\infty} a_k$.

Having a convergence regulator for a certain series, one can construct an FR-number which is the sum of this series.

Consider now the series

$$u_1 + \sum_{k=1}^{\infty} (u_{k+1} - u_k), \tag{1}$$

where u_k $(k \geq 1)$ is the R-integral of the function g over the system Ψ_k.

We shall say that the function g is **R-integrable with respect to the Zaslavskii sequence** if the series (1) converges. We shall say that g is **absolutely R-integrable with respect to the Zaslavskii sequence** if the series (1) converges absolutely.

The FR-number y will be called the **R-integral** of the function g, **taken with respect to the Zaslavskii sequence**, if y is the sum of the series (1).

Using property 2) of the sequence Φ and the elementary properties of the Riemann integral, it is not difficult to show that if g is Riemann integrable on $-\pi\triangle\pi$, and y is its Riemann integral over $-\pi\triangle\pi$, then g is R-integrable with respect to the Zaslavskii sequence and y is its R-integral taken with respect to the Zaslavskii sequence. It can also be shown that if g_1 is Riemann integrable on $-\pi\triangle\pi$, g_2 is nonnegative on $-\pi\triangle\pi$ and R-integrable with respect to the Zaslavskii sequence, then the product of these functions is absolutely R-integrable with respect to the Zaslavskii sequence.

Let f be an unbounded function continuous on the segment $-\pi\triangle\pi$, constructed according to Theorem 5.1 (^{1}). We shall also call this function the Zaslavskii function. From the construction of the Zaslavskii function the following properties are easily seen: 1) $f(x) = 0$ for $x \leq 0$ or $x \geq 1$ and $x \in -\pi\triangle\pi$; 2) $f(x) \geq 0$ for every x from $-\pi\triangle\pi$; 3) for every $n \geq 1$, at each point x belonging to the system Ψ_n, the inequality $f(x) \leq n$ holds; 4) for every FR-number from $-\pi\triangle\pi$ one can indicate a rational neighborhood of it in which the Zaslavskii function satisfies the Lipschitz condition.

Using property 3) of Zaslavskii' s function f and property 2) of the sequence Φ, one can show that f is absolutely R-integrable with respect to the Zaslavskii sequence.

5. Extend Zaslavskii' s function f from the segment $-\pi \leq x \leq \pi$ to the whole axis with period 2π, i.e., construct a function f_1 such that $f_1(x) = f(x)$ for $-\pi \leq x \leq \pi$ and $f_1(x+2\pi) = f_1(x)$ for every x. The function f_1 is not difficult to construct by using property 1) of the function f. (Incidentally,

such an extension can be carried out for any function g satisfying the condition $g(-\pi) = g(\pi)$.)

Theorem 1. *One can construct a function V that is a solution of the Dirichlet problem for the unit disk with boundary function f_1.*

Corollary. *There exists a function, continuous in the closed unit disk and harmonic inside the unit disk, that is unbounded in this disk.*

Lemma 1. *One can construct a function W, continuous in the closed unit disk and harmonic inside the unit disk, conjugate inside this disk to the function V constructed according to Theorem 1.*

The basic idea of the proof of Theorem 1 and Lemma 1 consists in using the known Poisson and Schwarz formulas, with the integration in the indicated formulas being performed with respect to the Zaslavskii sequence. In the proof of Lemma 1, property 4) of the function f is used.

From the corollary and Lemma 1 there follows the following theorem.

Theorem 2. *One can construct a c.f.c.f. φ, continuous in the closed unit disk, analytic inside the unit disk, and unbounded in this disk.*

Remark 1. Writing the Schwarz formula in complex form (see, for example, (5), p. 592), one can show that φ is holomorphic in the unit disk.

Remark 2. In connection with Theorem 1 the following question arises: can the Dirichlet problem for the unit disk be solved for every boundary constructive function? The same question is naturally posed under the condition that the boundary function be bounded. Both of these questions apparently remain open.

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* The function f_1 is here regarded as a function of the polar angle.

Note: Figure translations are in progress. See original paper for figures.