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Abstract

Full Text

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EMBEDDING NONORIENTABLE THREE-DIMENSIONAL MANIFOLDS IN FIVE-DIMENSIONAL EUCLIDEAN SPACE

(Presented by Academician L. S. Pontryagin, 14 VII 1964)

1. Formulation of the results.

In this note, by manifolds we mean smooth manifolds (without boundary or with boundary), by embeddings—smooth embeddings, and by isotopies—smooth isotopies (diffeotopies). Smoothness is understood to mean smoothness of class C^∞ .

The problem of embedding three-dimensional manifolds in Euclidean space R^5 was posed by Whitney in 1944 ⁽¹⁾, after he had proved the embeddability of manifolds of dimension n in R^{2n} ⁽²⁾. Since then the theory of embeddings has advanced far, and now we know that, as a rule, a manifold of dimension n can be embedded in R^{2n-1} . More precisely, in R^{2n-1} one can embed: every connected nonclosed manifold of dimension n ⁽³⁾; every orientable closed manifold of dimension $n \geq 5$ ^(4,5); every closed manifold of dimension $n \geq 5$, if n is not a power of two ⁽⁴⁾. As for closed three-dimensional manifolds, M. Hirsch proved ⁽⁶⁾ that every orientable closed three-dimensional manifold can be embedded in R^5 and even diffeomorphically to the boundary of a connected compact four-dimensional manifold embedded in R^5 .

The main result of the present paper is contained in the following theorem:

Theorem A. *Every nonorientable closed three-dimensional manifold can be embedded in R^5 .*

Thus every three-dimensional manifold can be embedded in R^5 .

The proof of Theorem A is based on two other embedding theorems concerning three-dimensional manifolds with boundary. Introduce in R^5 the usual coordinates x_1, \dots, x_5 , and agree to understand by R^4 and R^3 the subspaces of R^5 defined by the equations $x_5 = 0$ and $x_4 = x_5 = 0$, and by R_-^5 and R_+^5 the half-spaces defined by the inequalities $x_5 \leq 0$ and $x_5 \geq 0$. An embedding f of a three-dimensional manifold M in R^5 is called **special** if the manifold $f(M)$ lies in R_-^5 or in R_+^5 , intersects R^4 exactly along its boundary $\partial f(M) = f(\partial M)$, and nowhere touches R^4 . An embedding g of an orientable closed two-dimensional manifold P in R^4 is called **unknotted** if there exists a diffeomorphism $R^4 \rightarrow R^4$

taking the components of the manifold $g(P)$ to ordinary spheres with handles in R^3 . This definition is equivalent to each of the two following ones: the embedding $g : P \rightarrow R^4$ is unknotted if it is isotopic to an embedding taking the components of the manifold P to ordinary spheres with handles in R^3 ; the embedding $g : P \rightarrow R^4$ is unknotted if in R^4 there is a submanifold with boundary $g(P)$, diffeomorphic to a collection of ordinary bodies with handles. (That I \rightarrow III is obvious; that II \rightarrow I follows from the general theorem of Thom ^(7,8); that III \rightarrow II is easily proved.) Our auxiliary embedding theorems are as follows:

Theorem B. *Every orientable compact three-dimensional manifold M admits a special embedding in R^5 . Moreover, every unknotted embedding $\partial M \rightarrow R^4$ can be extended to a special embedding $M \rightarrow R^5$.*

Theorem C. *Every manifold M that decomposes into a smooth bundle of intervals over an orientable closed two-dimensional manifold admits an embedding in R^4 , unknotted on ∂M .*

2. Proof of Theorem B.

Let M be an orientable compact three-dimensional manifold and $f : \partial M \rightarrow R^4$ an unknotted embed-

tion. We may assume that the manifold $f(\partial M)$ lies in R^3 and bounds there a manifold M_1 composed of ordinary bodies with handles. Let M_2 be a second copy of the manifold M_1 , and let $g : M_1 \rightarrow M_2$ be a diffeomorphism. Glue M and M_2 by means of the diffeomorphism $gf : \partial M \rightarrow \partial M_2$ into an oriented closed manifold N , and construct, using the Hirsch theorem already mentioned, an embedding $h : N \rightarrow R^5$ such that $h(N)$ is the boundary of some simply connected submanifold V of the space R^5 . We may assume that, within some tubular neighborhood of the manifold $h(N)$, the manifold V consists of straight-line segments normal to $h(N)$. Let s be a displacement of the manifold $h(N)$ along these segments satisfying two conditions: $sh(M_2)$ lies on the boundary of the indicated tubular neighborhood; $sh(M)$ is a submanifold of this tubular neighborhood lying with its interior points inside it and orthogonal to its boundary at the points of its boundary $\partial sh(M) = \partial sh(M_2)$. Denote by W the complement (in R^5) of our tubular neighborhood, and consider the embedding $\alpha = shg : M_1 \rightarrow \partial W$.

Our aim is to extend the embedding α to a diffeomorphism $\varphi : R^5 \rightarrow R^5$ satisfying two conditions: $\varphi(R^5+) \subset W$; the manifold $\varphi(R^4)$ is tangent to the manifold ∂W along $\alpha(M_1)$. This will prove Theorem B: the embedding $\varphi^{-1}sh : M \rightarrow R^5$ satisfies all its requirements.

Let U be a ball in R^4 containing M_1 in its interior. First we extend the embedding $\alpha : M_1 \rightarrow \partial W$ to an embedding $\beta : U \rightarrow W$. This is possible, since the manifold W is connected and the homomorphisms of the fundamental groups of the components of the manifold M_1 into the group $\pi_1(W)$ corresponding to the embedding α are trivial (the latter follows from the simple-connectedness of the

manifold V). Obviously, the extension can be carried out so that the manifold $\beta(U)$ is tangent to the manifold ∂W along $\alpha(M_1)$, always on one and the same side, and so that the boundary of the manifold $\beta(U)$ lies inside W .

Now we must extend the embedding $\beta : U \rightarrow W$ to a diffeomorphism $\varphi : R^5 \rightarrow R^5$ satisfying the condition $\varphi(R^{5+}) \subset W$. The fact that β can be extended to *some* diffeomorphism $R^5 \rightarrow R^5$ follows from general theorems on the extension of diffeomorphisms⁽⁹⁾, and this simplifies our problem: we may assume that β is the identity transformation of the ball U and that the manifold ∂W is tangent to U on the side of R^{5-} . Let U_1 be a ball in R^4 concentric with U and larger than U , but still lying in W , and let Z be the cylinder over the ball U_1 in R^{5+} , i.e. the set of all points of all half-lines in R^{5+} issuing from the points of the ball U_1 normally to R^4 . Without loss of generality we may assume that $Z \subset W$. Indeed, from dimensional considerations it follows that in the half-space R^{5+} there are half-lines issuing from the center of the ball U_1 which do not lie in R^4 and do not intersect the manifold $h(N)$. By means of an affine transformation of the space R^5 , identical on R^4 , such a half-line can be made orthogonal to R^4 , and after a suitable shrinking of the neighborhood of the manifold $h(N)$, whose complement is W , this half-line turns out to lie in W together with some of its tubular neighborhood (the shrinking must not affect the neighborhood of the ball U_1). It remains to make the radius of the ball U_1 equal to the radius of the neighborhood of the half-line; this is achieved by a similarity transformation extended to a diffeomorphism of the space R^5 , identical outside the neighborhood of the ball U_1 . The cylinder Z is not a submanifold of the space R^5 in the sense of differential topology, since smoothness is violated on the boundary of the ball U_1 , but an ordinary smoothing (cutting off singularities), which can be performed without affecting U , turns Z into a submanifold. Obviously, β can be extended to a diffeomorphism $R^5 \rightarrow R^5$ which maps R^{5+} into this submanifold. We take this diffeomorphism to be φ .

3. Proof of Theorem C. It suffices to consider the connected case. From the general theorems of surface theory and bundle theory it fol-

gives that on orientable closed connected two-dimensional manifolds of the given genus p there exist, up to isomorphism, at most two smooth bundles of intervals: only one, the trivial one, if $p = 0$, and two, the trivial and the nontrivial one, if $p > 0$. Consequently, for $p = 0$ there exists only one class of diffeomorphic manifolds decomposing into such bundles, while for $p > 0$ there are only two classes, and it suffices to construct the required embedding for some one manifold of each class.

In the case of the trivial bundle, such an embedding is provided by the unit cylinder over the ordinary sphere with p handles, lying in R^3 (this cylinder consists of intervals of length 1 joining points of the indicated surface with the hyperplane $x_4 = 1$ of the space R^4). To pass to the nontrivial bundle, we make two transverse cuts on one of the handles of our surface, remove the cut-out tube, which is the product of a circle C and a unit interval I , remove together with this tube the part of the cylinder lying over it, which is the product $C \times I \times I$,

and glue this part of the cylinder back in again, but differently: one end we glue as before, by the identity diffeomorphism of the cylinder $C \times 0 \times I$, and the other by a diffeomorphism of the cylinder $C \times 1 \times I$ reflecting this cylinder in its midline $C \times 1 \times \frac{1}{2}$. In order to carry out this gluing in R^4 , we shall assume that $C \times 0 \times 0$ and $C \times 1 \times 0$ are unit circles in the planes $x_4 = 0$, $x_3 = 0$ and $x_4 = 0$, $x_3 = 1$, and that the cut-out tube $C \times I \times 0$ joined them rectilinearly. We obtain the required new position of the product $C \times I \times I$ in R^4 by turning the cylinder $C \times 0 \times I$ inside out in the three-dimensional subspace $x_3 = 0$ of the space R^4 (i.e., rotating the generators of this cylinder by 180° about the tangents to its midline) and simultaneously moving this subspace along the x_3 -axis into the position $x_3 = 1$. Of course, the whole operation must be carried out smoothly.

The boundary ∂M of the obtained three-dimensional manifold M is an orientable surface of genus $2p-1$, and it is easy to see that the embedding $\partial M \subset R^4$ is isotopic to the embedding carrying ∂M into the ordinary sphere with $2p-1$ handles in R^3 . Thus, the embedding $\partial M \subset R^4$ is unknotted.

4. Proof of Theorem A. It is enough to consider the connected case. Let M be a nonorientable connected closed three-dimensional manifold. Realize an element of the group $H_2(M; Z)$ having order 2 by an orientable closed two-dimensional submanifold Q , and denote by M_1 a tubular neighborhood of the submanifold Q , and by M_2 the closed complement of this neighborhood. It is clear that the manifold M_1 decomposes into a smooth bundle of intervals over Q , and that the manifold M_2 is orientable. By Theorem C, the manifold M_1 admits an embedding in R^4 unknotted on ∂M_1 , and a small deformation, identical on ∂M_1 , turns this embedding into a special embedding $\varphi_1 : M_1 \rightarrow R^5-$, such that the manifold $\varphi_1(M_1)$ at points of its boundary is C^∞ -orthogonal to R^4 . By Theorem B, the embedding $\varphi_1|_{\partial M_1} = \varphi_1|_{\partial M_2}$ can be extended to a special embedding $M_2 \rightarrow R^5+$, and a small deformation, identical on ∂M_2 , turns this embedding into such a special embedding $\varphi_2 : M_2 \rightarrow R^5+$ that the manifold $\varphi_2(M_2)$ at points of its boundary is C^∞ -orthogonal to R^4 . Together, φ_1 and φ_2 define an embedding $M \rightarrow R^5$.

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