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**Abstract**

**Full Text**

**L. I. KOVALENKO**

**A Difference Method and Uniqueness of a Generalized Solution for the Tricomi Problem**

*(Presented by Academician L. V. Kantorovich, 4 XII 1964)*

In Sec. 1 of this note, the difference method is applied to the solution of the Tricomi problem for an equation of mixed type. Along with convergence of the difference process, estimates are obtained for the solutions of the differential equation and of the corresponding system of difference equations, refining the estimates of papers <sup>(1,2)</sup> and generalizing the estimates of paper <sup>(3)</sup>. In contrast to <sup>(2)</sup>, all conditions are formulated on the coefficients of the differential equation that ensure convergence of the difference process and the validity of the indicated estimates.

In Sec. 2 the uniqueness of the generalized solution of the indicated problem is proved in the sense of an integral identity. The uniqueness of the smooth solution of the Tricomi problem and of the solution in the class  $R_1$  (see <sup>(4)</sup>) for equations of mixed type was obtained in papers <sup>(1,5,6)</sup>.

**1. A difference method for the Tricomi problem.** Let  $D$  be a domain situated in the strip  $-Y_1 \leq y \leq Y_2$ , ( $Y_1 > 0$ ,  $Y_2 > 0$ ), and bounded for  $y < 0$  by the characteristics  $\Gamma_0, \Gamma_A$  of the equation

$$Lu \equiv K(y)u''_{xx} + u''_{yy} + a(x, y)u'_x + b(x, y)u'_y + c(x, y)u = f(x, y) \quad (1)$$

$$(K(y) = |y|^\alpha q(y) \operatorname{sgn} y, \alpha > 0, q(y) > 0, q(\pm 0) > 0),$$

issuing from the points  $O(0, 0)$ ,  $A(x_1, 0)$ ,  $x_1 > 0$ , and for  $y \geq 0$  by a continuous curve  $\sigma$  with endpoints at the points  $O$  and  $A$ . We assume that  $O \in \sigma$ ,  $O \in \Gamma_0$ ,  $A \in \sigma$ ,  $A \in \Gamma_A$ . The set of points of the domain  $D$  for which  $y > 0$  ( $y < 0$ ) will be denoted by  $D^+$  ( $D^-$ ).

The Tricomi problem for equation (1) in  $\bar{D}$  ( $\bar{D}$  is the closure of  $D$ ) consists in finding a solution of equation (1), continuous in  $\bar{D}$  and satisfying the boundary condition

$$u = \varphi \quad \text{on } \Gamma_0 + \sigma, \quad (2)$$

where  $\varphi$  is a given continuous function.

Introduce the notation (for  $y < 0$ ):

$$\left[ \frac{d^m w}{dy^m} \right]_i = \frac{d^m}{dy^m} w \left[ x_0 - (-1)^i \int_y^0 \sqrt{-K(t)} dt, y \right], \quad \left[ \frac{dw}{dy} \right]_i \equiv \left[ \frac{d^1 w}{dy^1} \right]_i, \quad i, m = 1, 2;$$

$$d = 5 - \frac{4KK''}{(K')^2} - \left\{ 8aK'(-K)^{1/2} + 4a^2K + 8 \left[ \frac{da}{dy} \right]_2 (-K)^{3/2} - K^2 \left( 16c - 4b^2 - 8 \left[ \frac{db}{dy} \right]_2 \right) \right\} / K'^2;$$

$$g(w) = 4k [dw/dy]_1 (K' - 2a\sqrt{-K} + 2bK)^{-1}.$$

Let the coefficients of equation (1) satisfy the following conditions:

**Conditions I:** 1)  $K(y) \in C(\bar{D})$ ;  $K(y) \in C^{(3)}$  for  $-Y_1 \leq y < 0$ , and  $y^3 q'''$  is bounded,  $K' \geq M_1 |y|^{\alpha-1}$ ,  $M_1 = \text{const} > 0$ ; 2)  $a, b, c, f$  are continuous in  $\bar{D}$  for  $y \neq 0$ , and for  $y = 0$  only a discontinuity of the first kind is allowed;  $a, b \in C^{(2)}(D^-)$ , and  $y^m [d^m b/dy^m]_2, |y|^{m+1-\alpha/2} [d^m a/dy^m]_2$  ( $m = 1, 2$ ) are bounded in  $D^-$ ;  $|y|^{1-\alpha/2} a \rightarrow 0$  as  $y \rightarrow 0$  in  $\bar{D}$ ; 3)  $d > 0$  in  $D^-$  and is extended by continuity, with preservation of sign, to  $\bar{D}^-$ ; 4)  $c \leq 0$ ; 5)  $K' - 2a\sqrt{-K} + 2bK > 0$  on  $\Gamma_0$ .

Consider a mesh analogous to that introduced in <sup>(2,3)</sup>. For  $y \leq 0$  it consists of the points of intersection of the characteristics of equation (1) drawn through the division points of the segment  $OA$  into  $N = 2^{N_1}$  equal parts of length  $h$ ;  $(k, n)$  is the designation of the node  $(x, -y_n)$ ,  $x = kh + nh/2$ ,  $y_n = H(nh/2)$  ( $k, n = 0, 1, \dots$ ;  $y = -H(x)$  is the equation of the characteristic  $\Gamma_0$ ;  $y_0 = 0$ );  $h$  and  $l_n = y_n - y_{n-1}$  ( $n = 1, 2, \dots$ ) are the mesh steps in  $x$  and  $y$ . For  $y \geq 0$  the mesh is rectangular. It consists of points of the form  $x = kh$  ( $k = 0, \pm 1, \dots$ ),  $y = y'_n$  ( $n = 0, 1, \dots$ ), where  $y'_n = y_{2n}$ ;  $h$  is the step in  $x$ ,  $l'_n = l_{2n-1} + l_{2n}$  is the step in  $y$ . Denote by  $\bar{D}_h$  the set of all nodes belonging to  $\bar{D}$ ;  $\bar{D}_h$  is the mesh domain for  $\bar{D}$ . Introducing the notion of boundary as in <sup>(3)</sup>, denote the set of all boundary nodes for  $y \geq 0$  ( $y < 0$ ) by  $\sigma_h(\Gamma_{0h})$ , and  $\bar{D}_h - \sigma_h - \Gamma_{0h}$  by  $D_h$ . Let  $D_h^+$  ( $D_h^-$ ) be the totality of all nodes in  $D_h$  with  $y > 0$  ( $y < 0$ ), and  $\gamma_h$  those with  $y = 0$ .

For  $y < 0$ , to each node  $(k, n)$  from  $D_h^-$  we assign the equation, as in <sup>(2)</sup>,

$$\begin{aligned} R^- u_h \equiv & [(1 - A_{k,n})u_{k,n-1} + (1 + A_{k,n})u_{k-1,n+1} - (1 + \alpha_{k,n})u_{k-1,n} \\ & - (1 - \alpha_{k,n} - s_{k,n})u_{k,n}] / l_n l_{n+1} = f_{k,n}, \end{aligned} \quad (3)$$

where  $A_{k,n} \equiv a_n + B_{k,n}$ ,  $a_n \equiv (l_n - l_{n+1}) / (l_n + l_{n+1})$ ,  $B_{k,n} = -l_n(1 - a_n)b(x, -y_n)/2$ ,  $f_{k,n} \equiv f(x, -y_n)$ ,  $\alpha_{k,n} \equiv l_n l_{n+1} d(x, -y_n)/h$ ,  $s_{k,n} \equiv l_n l_{n+1} c(x, -y_n)$ . For  $y > 0$ , to each node  $(x, y'_n) \in D_h^+$  we assign the equation

$$K u_{hxx} + (1 + a'_n) u_{hy\bar{y}} + a(u_{hx} + u_{h\bar{x}}) / 2 +$$

$$+b[(1 - a'_n)u_{hy} + (1 + a'_n)u_{h\bar{y}}]/2 + cu_h = f. \quad (4)$$

All quantities in (4) are evaluated at the point  $(x, y'_n)$ ;  $a'_n = (l'_n - l'_{n+1})/(l'_n + l'_{n+1})$ ;  $u_{hx}, u_{h\bar{x}}, u_{hxx}$  are divided differences of the sought function  $u_h$  with respect to  $x$ , and  $u_{hy}, u_{h\bar{y}}, u_{hy\bar{y}}$  with respect to  $y$ , i.e.,  $u_{hy} = [u_h(x, y'_{n+1}) - u_h(x, y'_n)]/l'_{n+1}$ ,  $u_{h\bar{y}} = [u_h(x, y'_n) - u_h(x, y'_{n-1})]/l'_n$ ,  $u_{hy\bar{y}} = [u_{hy}(x, y'_n) - u_{hy}(x, y'_{n-1})]/l'_n$ . Difference equations (3) and (4) approximate equation (1), respectively for  $y < 0$  and  $y > 0$ .

To each node  $(x, 0) \in \gamma_h$  we assign the equation

$$[u_h(x, l'_1) - 2u_h(x, 0) + u_h(x, -l_1 - l_2)]/l_1^2 = 0. \quad (5)$$

Extend  $\varphi$  continuously to  $\bar{D}$  and denote the resulting function by  $\varphi_D$ . Let  $R$  be the difference operator acting according to the rule indicated in the left-hand sides of equalities (3)–(5), on any function defined in  $\bar{D}_h$ . We have the system of equations

$$Ru_h = f_h^* \quad \text{in } D_h, \quad u_h = \varphi_D \quad \text{on } \Gamma_{0h} + \sigma_h, \quad (6)$$

where  $f_h^* = f$  in  $D_h^+$  and  $D_h^-$ ;  $f_h^* = 0$  on  $\gamma_h$ . Existence and uniqueness of the solution of this system follow from Theorem 1.

**Theorem 1.** Suppose that

$$|a|h < 2K, \quad |b|l'_n < 2 \text{ in } D_h^+, \quad c \leq 0 \text{ in } D_h^+ + D_h^-; \quad (7)$$

$$|A_{k,n}| \leq 1, \quad \alpha_{k,n} + s_{k,n} < 1, \quad A_{1,n} > \alpha_{1,n} \\ \text{for } k, n = 1, 2, \dots; \quad (8)$$

$$(1 + A_{k+1,n-1})(1 - A_{k,n}) \geq (1 - \alpha_{k,n} - s_{k,n})(1 + \alpha_{k+1,n-1}) \\ \text{for } k = 1, 2, \dots; \quad n = 2, 3, \dots \quad (9)$$

Let the function  $w_h$  be defined in  $\bar{D}_h$ ;  $Rw_h \leq 0$  in  $D_h$ ;  $w_h \geq 0$  on  $\sigma_h$ ,

$$w_{0,n} \geq 0, \quad (1 + A_{1,n})w_{0,n+1} \geq (1 + \alpha_{1,n})w_{0,n} \quad (n = 1, 2, \dots). \quad (10)$$

Then  $w_h \geq 0$  in  $\bar{D}_h$ .

From Theorem 1 there follows a maximum principle more general than that formulated in (2), Theorem 3.3.

For the proof of Theorem 1 the following is important.

**Lemma 1.** Suppose that conditions (8), (9) are satisfied. Let the function  $w_h$  be defined in  $\overline{D}_h - (0, 0) - (x_1, 0)$ ,  $R^-w_h \leq 0$  in  $D_h^-$ , and, for  $y = 0$ ,  $w_h \geq 0$ , while on  $\Gamma_{0h}$  (10) holds.

Then  $w_h \geq 0$  and

$$(1 + A_{k+1,n})w_{k,n+1} \geq (1 + \alpha_{k+1,n})w_{k,n}$$

for  $k = 0, 1, \dots; n = 1, 2, \dots$

**Theorem 2.** Suppose that, in addition to (7)–(9), we have

$$A_{1,n} - \alpha_{1,n} \geq C_* l_{n+1} \quad \text{for } n = 1, 2, \dots, \quad (11)$$

where  $C_* = \text{const} > 0$ . Then for the solution  $u_h$  of system (6), for sufficiently small  $h$ , the estimate holds

$$\begin{aligned} \min\{0, \varphi_h, \psi_n - (1 + A_{1,n})(\psi_n - \psi_{n+1})/(A_{1,n} - \alpha_{1,n})\} - k_2 M &\leq u_h \leq \\ &\leq \max\{0, \varphi_h, \psi_n - (1 + A_{1,n})(\psi_n - \psi_{n+1})/(A_{1,n} - \alpha_{1,n})\} + k_1 M, \end{aligned} \quad (12)$$

where  $\varphi_h = \varphi_D$  on  $\sigma_h + \Gamma_{0h}$ ,  $\psi_n = \varphi(nh/2, -y_n)$ ,  $M = \text{const} > 0$ , and the numbers  $k_1, k_2 \geq 0$  are such that  $-k_1 \leq f \leq k_2$  in  $D$ .

If  $c \equiv 0$ , then the zero under the signs min and max in (12) may be omitted.

The method of proof of Lemma 1 and Theorems 1 and 2 is analogous to the method of proof of, respectively, Lemmas 3 and 5 and Theorem 3 in (2).

Conditions I ensure the validity of (7)–(9), (11) for sufficiently small  $h$ . In (2) inequality (9) was proved only for  $y \leq -\delta < 0$ .

By the usual device one proves the convergence of  $u_h$  to  $u$  as  $h \rightarrow 0$  under the assumption that  $u$  is a solution of equation (1) possessing derivatives  $u'_x, u'_y, u''_{xx}$ , which are continuous in  $\overline{D}$ , and  $u''_{yy}$ , which is continuous in  $\overline{D}$  for  $y \neq 0$  and, for  $y = 0$ , may have only a discontinuity of the first kind.

For  $u$  the estimate holds

$$\min\{0, \varphi, \psi + g_0(\psi)\} - k_2 M \leq u \leq \max\{0, \varphi, \psi + g_0(\psi)\} + k_1 M, \quad (13)$$

where  $\psi = \varphi$  on  $\Gamma_0$ ,  $g_0(\psi) = g(\psi)$  on  $\Gamma_0$ ;  $M, k_1, k_2$  are the same as in (12). If  $c \equiv 0$ , then the zero under the signs min and max in (13) may be omitted. Inequality (13) is obtained from (12) by using the convergence of  $u_h$  to  $u$ .

## 2. Uniqueness of the generalized solution of the Tricomi problem

Let  $v$  be a function defined in  $\bar{D}$  and having the following properties: 1)  $v$  is continuous in  $\bar{D}$ ; 2)  $v = \varphi$  on  $\Gamma_0 + \sigma$ ; 3)  $v$  satisfies a Lipschitz condition in  $x$  and  $y$  in any closed domain  $\bar{D}_1$  contained in  $D - \sigma$ ; 4) for any  $\Phi \in C^{(1)}(D)$ , equal to zero in the boundary strip,

$$\iint_D [K\Phi'_x v'_x + \Phi'_y v'_y + (f - av'_x - bv'_y - cv)\Phi] dx dy = 0; \quad (14)$$

- 5)  $g(v)$  is uniformly continuous on  $\Gamma_0$  with respect to the set on which  $[dv/dy]_1$  exists.

We shall call such a function  $v$  a generalized solution of the Tricomi problem for equation (1) with boundary condition (2). A function possessing properties 3), 4) will be called a generalized solution of equation (1) in  $D$ . The generalized solution of equation (1) in any domain  $D_2$ ,  $D_2 \subset D$ , is defined analogously.

We assume that the coefficients of equation (1) satisfy, in addition to conditions I (see § 1), the following conditions:

**Conditions II:** 1) the function  $a(x, y)$ , extended by continuity to  $y = 0$  from the side  $y > 0$  ( $y < 0$ ), has a continuous  $a'_x$  for  $y \geq 0$  ( $y \leq 0$ ) in  $D$ ; 2)  $b \in C^{(1)}(D)$  and has a continuous  $b''_{yy}$  in  $D$ .

Denote by  $\{v\}_r$  the averaging of the function  $v$  with respect to  $x$  with radius  $r$  (see (7)).

**Theorem 3.** *Let  $w$  be a generalized solution of equation (1) in  $D$ . Then in any closed subdomain  $\bar{D}_2$  of the domain  $D$ ,  $\{w\}_r \rightarrow w$  as  $r \rightarrow 0$ , and, for  $y \neq 0$ , for sufficiently small  $r$  we have  $L\{w\}_r = f^{(r)}(x, y; r)$ , where*

$$f^{(r)} = a \partial \{w\}_r / \partial x - \partial \{aw\}_r / \partial x + b \partial \{w\}_r / \partial y - \partial \{bw\}_r / \partial y + c \{w\}_r + \{(\partial a / \partial x + \partial b / \partial y - c)w + f\}_r;$$

$f^{(r)}$  is continuous in  $\bar{D}_2$  for  $y \neq 0$  and is extended by continuity to  $y = 0$  from the side  $y > 0$  ( $y < 0$ ), if  $\bar{D}_2$  has points with  $y = 0$  and  $y > 0$  ( $y < 0$ );  $f^{(r)} \rightarrow f$  as  $r \rightarrow 0$  in  $\bar{D}_2$  for  $y > 0$  ( $y < 0$ ).

The existence of continuous  $\partial^m \{w\}_r / \partial x^m$  is obvious. The existence of a continuous  $\partial^2 \{w\}_r / \partial y^2$  for  $y \neq 0$  can be verified by showing that the function

$$\theta(x, y; r) = -K \partial \{w'_x\}_r / \partial x - \{aw'_x + bw'_y + cw - f\}_r$$

is the generalized derivative with respect to  $y$  of  $\{\partial w / \partial y\}_r$ , and that for  $y \neq 0$ ,  $\theta$  is equivalent to a continuous function.

Having proved that  $z = \{w\}_r$ , for sufficiently small  $r$ , is a generalized solution of the equation  $Lz = f^{(r)}$  in  $D_2$ , and taking into account the continuity of the derivatives of  $\{w\}_r$  entering this equation, we obtain that  $\{w\}_r$  satisfies it in the ordinary sense.

**Theorem 4 (uniqueness theorem).** *Let  $v$  be a generalized solution of the Tricomi problem for equation (1), and suppose  $v = 0$  on  $\Gamma_0 + \sigma$ . Then  $v \equiv 0$  in  $\bar{D}$ .*

To prove Theorem 4 we consider the averages  $\{v\}_r$  and apply estimate (13) to them. From it and from the smallness of  $|v - \{v\}_r|$  it follows that  $v \equiv 0$  in  $\bar{D}$ .

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