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Abstract

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MATHEMATICS

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ON STABLE SOLVABLE EXTENSIONS OF PARTIAL DIFFERENTIAL OPERATORS

(Presented by Academician V. I. Smirnov on 5 I 1965)

In §§ 1 and 2 of the present paper, for a general partial differential operator with constant coefficients in a bounded domain, a criterion is established for the stability of a correct homogeneous boundary-value problem with respect to small perturbations of the coefficients of the differential expression. In § 3, stability with respect to small, in a certain sense, perturbations of the boundary conditions is considered. The results obtained here are also valid for operators with variable coefficients.

1. Let, in an n -dimensional bounded domain Ω , a differential expression be given

$$\mathcal{P}(\mathcal{D}) = \sum_{|\alpha| \leq m} a_{\alpha_1 \dots \alpha_n} \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_n^{\alpha_n} = \sum_{|\alpha| \leq m} a_{\alpha} \mathcal{D}^{\alpha} \quad (1)$$

with complex constant coefficients, and let P_0, P be, respectively, the minimal and maximal operators generated by it in $L_2(\Omega)$. Let \hat{P} be a solvable extension (s.e.) of the operator P_0^* ; let $\mathcal{Q}(\mathcal{D})$ be another differential expression of the form (1), of order not exceeding m ; let $(P + \lambda Q)_0$ be the minimal operator and $P + \lambda Q$ the maximal operator generated in $L_2(\Omega)$ by the expression $\mathcal{P}(\mathcal{D}) + \lambda \mathcal{Q}(\mathcal{D})$ (λ is a complex constant).

Definition. The s.e. \hat{P} is called Q -stable if there exists a number $\varepsilon > 0$ such that, for $|\lambda| < \varepsilon$, the following conditions are satisfied:

- 1) $D(\hat{P}) \subset D(P + \lambda Q)$; 2) the restriction $\widehat{P + \lambda Q}$ of the operator $P + \lambda Q$ to $D(\hat{P})$ is an s.e. of the operator $(P + \lambda Q)_0$.

Theorem 1. *The operator P_0 has at least one Q -stable s.e. if and only if*

$$|\mathcal{Q}(\xi)|^2 \leq C \sum_{\alpha} |\partial^{|\alpha|} \mathcal{P} / \partial \xi_1^{\alpha_1} \dots \partial \xi_n^{\alpha_n}|^2 \quad (2)$$

for all real $\xi = (\xi_1, \dots, \xi_n)$ ($C > 0$ is a constant).

In the proof of Theorem 1, the differential operators are compared by the method of L. Hörmander ⁽²⁾, and the construction, given in his monograph ⁽³⁾, of a fundamental solution for an operator with constant coefficients is used.

We shall say that the differential expression $\mathcal{P}(\mathcal{D})$ is stable with respect to the coefficient $a_\alpha = a_{\alpha_1 \dots \alpha_n}$ if the minimal operator P_0 has at least one \mathcal{D}^α -stable s.e.

The following examples illustrate the possibilities of applying Theorem 1**.

* This means ⁽¹⁾ that $P_0 \subset \hat{P} \subset P$ and that there exists a bounded operator \hat{P}^{-1} defined on the whole space $L_2(\Omega)$.

** We exclude from consideration differential monomials that are obviously stable with respect to their nonzero coefficient.

- 1) In order that $\mathcal{P}(\mathcal{D})$ be stable with respect to all coefficients (including those equal to zero) corresponding to $|\alpha| \leq m$, it is necessary and sufficient that it be elliptic.
- 2) If $\mathcal{P}(\mathcal{D})$ is parabolic in the sense of I. G. Petrovsky ⁽⁴⁾ or 2b-parabolic in the sense of S. D. Eidelman ⁽⁵⁾, then it is stable with respect to all nonzero coefficients. Operators parabolic in the sense of G. E. Shilov ⁽⁶⁾ may no longer have this property. Thus, the expression $\partial/\partial x_1 - \partial^2/\partial x_2^2 - \partial^3/\partial x_2^3$ is stable with respect to the coefficient at $\partial^2/\partial x_2^2$, but is not stable with respect to the coefficients at $\partial/\partial x_1$ and $\partial^3/\partial x_2^3$.
- 3) The pseudoparabolic expression $\partial/\partial x_1 - \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ is not stable with respect to the coefficients at $\partial^2/\partial x_2^2$ and $\partial^2/\partial x_3^2$, and is stable with respect to the coefficient at $\partial/\partial x_1$ and the coefficients equal to zero at $\partial/\partial x_2$ and $\partial/\partial x_3$.
- 4) The expression $\partial/\partial x_1 - i\partial^2/\partial x_2^2$ is not stable with respect to any nonzero coefficient and is stable with respect to the coefficient equal to zero at $\partial/\partial x_2$. The stability of the hyperbolic expression $\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \dots - \partial^2/\partial x_n^2$ and of the ultrahyperbolic expression $\partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_k^2 - \partial^2/\partial x_{k+1}^2 - \dots - \partial^2/\partial x_n^2$ has the same character.

Remarks. 1) Some of the ultrahyperbolic and pseudoparabolic operators with r.r. constructed in the work of A. A. Dezin ⁽⁷⁾ are stable in the sense indicated there and with respect to small perturbations of the coefficients at the highest derivatives. This occurs because, in the approach to the stability problem developed in ⁽⁷⁾, the domains of definition of the original and the perturbed r.r. may turn out to be different. Our definition of stability excludes this possibility.

- 2) For operators with real coefficients that are stronger than the Laplace operator, another definition of stability was given by P. P. Mosolov ⁽⁸⁾. Polynomials stable in Mosolov's sense are stable in our sense with re-

spect to positive coefficients in terms containing only even powers of the variables.

2. If \hat{P} and \tilde{P} are two r.r. of the operator P_0 , then, as established by M. I. Vishik ⁽¹⁾,

$$D(\tilde{P}) = D(P_0) + (\hat{P}^{-1} + K)V, \quad (3)$$

where $V = \{v(x) : v \in L_2(\Omega), \bar{P}v = 0\}$ ($\bar{P}(\mathcal{D})$ is the expression adjoint to $\mathcal{P}(\mathcal{D})$); K is a bounded operator, $D(K) = V$, $R(K) \subset U$; $U = \{u(x) : u \in L_2(\Omega), Pu = 0\}$.

Theorem 2. The Q -stability of an r.r. \hat{P} entails the Q -stability of the r.r. \tilde{P} , described by relation (3), if and only if $R(K) \subset \bar{D}(Q)$.

3. Let

$$\mathcal{P}(x, \mathcal{D}) = \sum_{|\alpha| \leq m} a_\alpha(x) \mathcal{D}^\alpha$$

be a differential expression with complex-valued variable coefficients, and let P_0, P be the minimal and maximal operators generated by it in $L_2(\Omega)$. We shall assume that P_0 has an r.r. \hat{P} , and let $D(\hat{P}) = \hat{D}$. Consider the Hilbert space H_P of elements $g(x) \in D(P)$ with scalar product $[g, h] = (g, h) + (Pg, Ph)$, where $(,)$ is the scalar product in $L_2(\Omega)$. We denote the norm in H_P by $|||$, and the norm in $L_2(\Omega)$ by $\| \|$. The space H_P decomposes into the direct sum of the subspaces \hat{D} and U (cf. ^(1,2)). Denote by χ the projection operator projecting H_P onto U parallel to \hat{D} .

Let \tilde{D} ($D(P_0) \subset \tilde{D} \subset D(P)$) be another subspace in H_P , and

$$\|\chi\psi\| \leq \delta\|\psi\|, \quad \forall \psi \in \tilde{D}, \quad (4)$$

where δ ($0 < \delta < 1$) is a constant.

Theorem 3. In order that the restriction \tilde{P} of the operator P to the subspace \tilde{D} satisfying condition (4) be an r.r. operator P_0 , it is necessary and, for $\delta < 1/2$, sufficient that from the relations

$$\varphi \in \hat{D}, \quad [\varphi, \psi] = 0, \quad \forall \psi \in \tilde{D} \quad (5)$$

it follow that $\varphi = 0$.*

The application of Theorem 3 to boundary-value problems described by means of a system of homogeneous boundary conditions proceeds according to the following scheme.

Denote by Γ the boundary of the domain Ω , and let $H(\Gamma)$ be some Hilbert space of vector-valued functions defined on Γ , with scalar product $\langle \cdot, \cdot \rangle$. Suppose that for all $u(x), v(x) \in C^\infty(\bar{\Omega})$ the representation

$$(\mathcal{P}u, v) - (u, \bar{\mathcal{P}}v) = \langle \mathcal{A}u|_\Gamma, \mathcal{B}v|_\Gamma \rangle + \langle \mathcal{B}'u|_\Gamma, \mathcal{A}'v|_\Gamma \rangle = \langle \tilde{\mathcal{A}}u|_\Gamma, \tilde{\mathcal{B}}v|_\Gamma \rangle + \langle \tilde{\mathcal{B}}'u|_\Gamma, \tilde{\mathcal{A}}'v|_\Gamma \rangle, \quad (6)$$

is valid, where $\mathcal{A}, \mathcal{B}, \dots, \mathcal{A}'$ are matrix linear differential expressions of order not exceeding $m - 1$ in $C^\infty(\Gamma)$. Let \widehat{D} be the closure in H_P of the set of functions $\varphi(x) \in C^\infty(\bar{\Omega})$ for which $\mathcal{A}\varphi|_\Gamma = 0$; let \widetilde{D} be the closure in H_P of the set of functions $\psi(x) \in C^\infty(\bar{\Omega})$ for which $\tilde{\mathcal{A}}\psi|_\Gamma = 0$; \widehat{P} and \widetilde{P} are the restrictions of the operator P to the subspaces \widehat{D} and \widetilde{D} , respectively. We shall also assume that the operator \widetilde{P}^* , adjoint to \widetilde{P} with respect to the scalar product (\cdot, \cdot) , can be defined as the closure in $L_2(\Omega)$ of the differential expression $\widetilde{P}(D)$, given on functions $v(x) \in C^\infty(\bar{\Omega})$ for which $\tilde{\mathcal{A}}'v|_\Gamma = 0$.

Let the boundary values of all elements of $\widehat{D}, \widetilde{D}$, and $D(\widetilde{P}^*)$ belong to $H(\Gamma)$, and let the differential expressions $\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{B}, \tilde{\mathcal{B}}, \mathcal{A}'$ be extendable to operators $A, \tilde{A}, \tilde{B}, A'$, respectively, acting in $H(\Gamma)$, so that

$$A\varphi|_\Gamma = 0, \quad \forall \varphi \in \widehat{D}; \quad \tilde{A}\psi|_\Gamma = 0, \quad \forall \psi \in \widetilde{D}; \quad \tilde{A}'v|_\Gamma = 0, \quad \forall v \in D(\widetilde{P}^*);$$

$$(P\varphi, v) - (\varphi, \widetilde{P}v) = \langle \tilde{A}\varphi|_\Gamma, \tilde{B}v|_\Gamma \rangle, \quad \forall \varphi \in \widehat{D}, \quad \forall v \in D(\widetilde{P}^*).$$

We require, in addition, that the following conditions be fulfilled:

- 1) $\langle A\psi|_\Gamma, A\psi|_\Gamma \rangle \leq \eta_1 \|\psi\|^2, \quad \forall \psi \in \widetilde{D}$ ($\eta_1 > 0$ is a constant);
- 2) $|\langle \tilde{A}\varphi|_\Gamma, \tilde{B}v|_\Gamma \rangle| \leq \eta_2 \|\varphi\| \|v\|, \quad \forall \varphi \in \widehat{D}, \quad \forall v \in D(\widetilde{P}^*)$ ($\eta_2 > 0$ is a constant; $\|v\|^2 = \|v\|^2 + \|\widetilde{P}v\|^2$);
- 3) $\|u\|^2 \leq a \langle Au|_\Gamma, Au|_\Gamma \rangle, \quad \forall u \in U$, representable in the form $u = \psi - \varphi, \quad \psi \in \widetilde{D}, \quad \varphi \in \widehat{D}$ ($a > 0$ is a constant).

Theorem 4. Suppose that all the conditions listed above are fulfilled and that \widehat{P} is an r.r. operator P_0 . If $a\eta_1 < 1/2$ and $\eta_2 < 1$, then \widetilde{P} is also an r.r. operator P_0 .

According to the indicated scheme one can resolve the question of the stability of correct problems, both elliptic and non-elliptic (for example, those considered in (7)). Here \widetilde{P} is Q -stable if and only if $\|Q\psi\|^2 \leq C\|\psi\|^2, \quad \forall \psi \in \widetilde{D}$ ($C > 0$ is a constant).

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* In the proof of Theorem 3 the notion of a regular pair of subspaces introduced in (9) is used.

Note: Figure translations are in progress. See original paper for figures.

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