Reports of the Academy of Sciences of the USSR

1965

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1965. Volume 164, No. 2

MATHEMATICS

Yu. V. KOMLENKO

ON SOME CRITERIA FOR NONOSCILLATION AND BOUNDEDNESS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

(Presented by Academician L. S. Pontryagin on 15 III 1965)

1. Consider the problem

$$x^{(n)} + p(t)x = 0, \tag{1}$$

$$x^{(k)}(0) = 0, \quad k = 0, 1, \dots, n-2, \quad x^{(n-1)}(0) = 1, \tag{2}$$

where $p(t)$ is a real function of the real variable t, summable on every finite interval. The present paper is devoted to estimating the length of the interval $(0, R)$ on which the solution $x(t)$ of problem (1), (2) has no zeros. To this estimate are reduced questions concerning the interval of applicability of Chaplygin' s theorem ([10,14]), the conditions for preservation of the sign of the Green' s function for a number of boundary-value problems for equation (1) ([10,14,16,17]), and also the problem of boundedness of solutions of equation (1) for $n = 2$ with a periodic coefficient ([1,2]).

An effective estimate of the length of the interval mentioned may be obtained on the basis of certain theorems on integral inequalities from the works ([10,11,14,15]) of the Izhevsk seminar. The idea of applying these theorems is that, for some positive comparison function $v(t)$, the theorems mentioned guarantee the inequality $v(t) \leqslant x(t)$.

On the basis of the "fork" theorem of ([15]) we have the following generalization of a number of assertions of ([10,11,13,14]).

Put $K(t, s) = g(t, s)h(s)$, where $g(t, s) \geqslant 0$ and is continuous for $a \leqslant s \leqslant t < b$; $h(s)$ is summable on every finite interval belonging to $[a, b)$, and $h(s) \leqslant 0$ almost everywhere.

Lemma. If $f(t)$ is continuous in $[a,b)$, then in $[a,b)$ the integral inequality

$$v(t) = f(t) + \int K(t,s)f(s)\,ds \geqslant 0$$

guarantees the estimate

$$v(t) \leqslant u(t) \leqslant f(t)$$

for the solution $u(t)$ of the equation

$$u(t) = f(t) + \int_a^t K(t,s)u(s)\,ds.$$

Reducing, in one way or another, problem (1), (2) to a Volterra equation and using the lemma, one can obtain an estimate of the length of the interval $(0,R)$ free of zeros of the solution of the problem under consideration. Such estimates are given below, using the following comparison theorem ([7,10]): if $x(t)$ is the solution of problem (1), (2), and $y(t)$ is the solution of the equation $y^{(n)} + p_+(t)y = 0$ with the initial conditions (2), and $y(t) > 0$ in $(0,R)$, then $y(t) \leqslant x(t)$ on this interval. (Here and below $p_+(t) = \frac{1}{2}[|p(t)| + p(t)]$.)

Let $C(t,s)$ be the Cauchy function ([9]) of the operation $x^{(n)} + f(t)x$ with a coefficient summable on every finite interval:

$$C(t,s) = \begin{vmatrix} u_0(s) & \dots & u_{n-1}(s) \\ \cdot & \dots & \cdot \\ u_0^{(n-2)}(s) & \dots & u_{n-1}^{(n-2)}(s) \\ u_0(t) & \dots & u_{n-1}(t) \end{vmatrix} : \begin{vmatrix} u_0(s) & \dots & u_{n-1}(s) \\ \cdot & \dots & \cdot \\ u_0^{(n-2)}(s) & \dots & u_{n-1}^{(n-2)}(s) \\ u_0^{(n-1)}(s) & \dots & u_{n-1}^{(n-1)}(s) \end{vmatrix},$$

where $u_k(t)$ $(k = 0,1,\dots,n-1)$ is a fundamental system of solutions of the equation $x^{(n)} + f(t)x = 0$. It is not difficult to see that problem (1), (2) reduces to the equation

$$x(t) = \int_0^t C(t,s)[f(s) - p(s)]x(s)\,ds + C(t,0).$$

Theorem 1. If in the triangle $0 < s < t < R$

$$C(t,s) > 0, \qquad \int_s^t C(t,r)C(r,s)[p(r) - f(r)]_+\,dr < C(t,s),$$

then in the interval $(0,R)$ the solution of problem (1), (2) has no zeros.

On the basis of Theorem 1 we obtain the following assertions.

Theorem 2. If almost everywhere on $[0, R]$ the inequality $p(t) \le (2n-1)!/(n-1)!R^n$ holds, then in the interval $(0, R)$ the solution of problem (1), (2) has no zeros.

Theorem 3. Suppose that almost everywhere on $[0, R]$ $p(t) \le C_n(\pi/R)^n$ $(n = 2, 3, 4)$, where $C_2 = 1,\ C_3 = 343/81\sqrt{3},\ C_4 = 9$. Then in $(0, R)$ the solution of problem (1), (2) has no zeros. If, however, $|p(t)| \le C_3(\pi/R)^3$ almost everywhere on $[0, R]$, then a nontrivial solution of equation (1) for $n = 3$ has in $[0, R]$ no more than two zeros, counting a multiple zero twice.

Remark. The estimates given in Theorems 2 and 3 are sharper than the estimates following from the work ([12]).

2. Consider the equation

$$x'' + p(t)x = 0, \tag{3}$$

where $p(t)$ is a function summable on every finite interval and not equivalent to a constant.

An interval $[0, R]$ in which a nontrivial solution of equation (3) has no more than one zero will be called a **nonoscillation interval** ([10,14]).

Let $\lambda = \text{const}$,

$$C(t, s) = \begin{cases} \dfrac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(t-s), & \lambda > 0, \\ t - s, & \lambda = 0, \\ \dfrac{1}{\sqrt{|\lambda|}} \operatorname{sh} \sqrt{|\lambda|}(t-s), & \lambda < 0. \end{cases}$$

From Theorem 1 it follows:

Theorem 4. If for some $\lambda \le \pi^2/R^2$

$$\int_0^R C(R, s)C(s, 0)[p(s) - \lambda]_+ \, ds \le C(R, 0), \tag{4}$$

then $[0, R]$ is a nonoscillation interval.

Remark. Inequality (4) is satisfied if

$$\int_0^R [p(t)-\lambda]_+\,dt \le \begin{cases} 2\sqrt{\lambda}\ \mathrm{ctg}\, R\sqrt{\lambda}/2, & 0<\lambda\le \pi^2/R^2,\\ \dfrac{4}{R}, & \lambda=0,\\ 2\sqrt{|\lambda|}\ \mathrm{cth}\, R\sqrt{|\lambda|}/2, & \lambda<0. \end{cases} \tag{5}$$

3. We shall further assume that $p(t+\omega)=p(t)$, $\omega>0$. From Floquet theory it is known that if $u_0(t), u_1(t)$ are solutions of equation (3), with $u_0(0)=0,\ u_0'(0)=1,\ u_1(0)=1,\ u_1'(0)=0$, then from the inequality $|u_0'(\omega)+u_1(\omega)|<2$ the boundedness of the solutions of equation (3) follows. Estimating, with the aid of a lemma, the sum $u_0'(t)+u_1(t)$, one can obtain boundedness criteria for the solutions. (In ([10]) it is shown that the estimate of this sum

on the basis of the theorem on differential inequalities leads to an elementary proof of Lyapunov' s criterion.) However, if one uses the results of papers ([1], [2]), then boundedness criteria can be obtained on the basis of estimates of the length of the nonoscillation interval.

Theorem 5. Let

$$\int_0^\omega p(t)\,dt \ge 0.$$

If for some $\lambda\le \pi^2/\omega^2$ and some $a\in[0,\omega]$

$$\int_a^{a+\omega} C(a+\omega,t)C(t,a)[p(t)-\lambda]_+\,dt \le C(a+\omega,a), \tag{6}$$

where the equality sign is allowed only when $\lambda=\pi^2/\omega^2$, then all solutions of equation (3) are bounded.

Remark. Inequality (6) is satisfied if inequality (5) is satisfied with $R=\omega$.

Denote

$$F(\lambda,n)=\begin{cases} \dfrac{\omega\sqrt{\lambda}}{2(n+1)}\ \mathrm{ctg}\,\dfrac{\omega\sqrt{\lambda}}{2(n+1)}, & 0<\lambda\le \dfrac{(n+1)^2\pi^2}{\omega^2},\\ 1, & \lambda=0,\\ \dfrac{\omega\sqrt{|\lambda|}}{2(n+1)}\ \mathrm{cth}\,\dfrac{\omega\sqrt{|\lambda|}}{2(n+1)}, & \lambda<0. \end{cases}$$

Theorem 6. If for some $n=0,1,\ldots$ and some $\lambda\le (n+1)^2\pi^2/\omega^2$

$$p(t) \geq \frac{n^2\pi^2}{\omega^2}, \qquad \int_0^\omega [p(t)-\lambda]_+ \, dt \leq \frac{4(n+1)^2}{\omega} F(\lambda, n)$$

(the first inequality is understood almost everywhere in $[0,\omega]$), then all solutions of equation (3) are bounded.

Remark 1. From Theorem 5, as a special case (as applied to the zero stability zone O_0 in the sense of Yakubovich ([4])), we obtain, for

$$\lambda = \frac{1}{\omega}\int_0^\omega p(t)\, dt,$$

an improvement of one of G. Borg' s results ([3]).

Remark 2. From Theorems 5 and 6 there follow certain stability criteria given in papers ([1], [4]-[6]).

Remark 3. If one sets $p(t) = \lambda + f(t)$, then from Theorem 6 there easily follows an improvement of a number of results of K. R. Putnam' s paper ([8]) on the estimation of stability intervals of the equation $x'' + (\lambda + f(t))x = 0$.

In conclusion, the author expresses his deep gratitude to Prof. N. V. Azbelev for suggesting the topic and to Prof. V. A. Yakubovich for his attention to the work.

Received
4 III 1965

REFERENCES

[1] N. E. Zhukovskii, *Matem. sborn.*, **16**, No. 3 (1892).
[2] N. V. Adamov, *DAN*, **2**, No. 5-6 (1935).
[3] G. Borg, *Arch. för mat., astr. och fysik*, **31A**, No. 1 (1944).
[4] V. A. Yakubovich, *DAN*, **74**, No. 5 (1950).
[5] M. G. Krein, *Prikl. matem. i mekh.*, **15**, No. 3 (1951).
[6] V. A. Yakubovich, *Prikl. matem. i mekh.*, **18**, No. 6 (1954).
[7] N. V. Azbelev, Z. B. Tsalyuk, *Ukr. matem. zhurn.*, **10**, No. 1 (1958).
[8] C. R. Putnam, *Quart. Appl. Math.*, **16**, No. 4 (1959).
[9] N. V. Azbelev, I. M. Smolin, Z. B. Tsalyuk, *DAN*, **135**, No. 3 (1960).
[10] N. V. Azbelev, Doctoral dissertation, Kazan, 1962.
[11] N. V. Azbelev, Z. B. Tsalyuk, *Matem. sborn.*, **56**, No. 3 (1962).
[12] G. A. Bessmertnykh, A. Yu. Levin, *DAN*, **144**, No. 3 (1962).
[13] S. A. Pak, *Sibirsk. matem. zhurn.*, **3**, No. 4 (1962).
[14] N. V. Azbelev, Z. B. Tsalyuk, Proceedings of the IV Mathematical Congress, 2, "Nauka," 1964.
[15] Yu. V. Komlenko, Reports of the III Siberian Conference on Mathematics and Mechanics, Tomsk, 1964.

[16] V. V. Ostroumov, ibid.
[17] V. A. Churikov, ibid.

Note: Figure translations are in progress. See original paper for figures.