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**Abstract**

**Full Text**

**MATHEMATICS**

**B. A. VOSTRETSOV**

## ON THE STRUCTURE OF ANALYTIC SOLUTIONS OF ONE CLASS OF SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

*(Presented by Academician I. G. Petrovskii on 16 XI 1964)*

Let  $l_1(\partial/\partial x), \dots, l_k(\partial/\partial x)$  be symbolic forms with respect to  $\partial/\partial x_1, \dots, \partial/\partial x_n$  with constant coefficients. In the paper the structure of the solutions of the system

$$l_1(\partial/\partial x)u = 0, \dots, l_k(\partial/\partial x)u = 0, \quad (1)$$

analytic in a prescribed domain  $D$  of the  $n$ -dimensional space  $R_n = \{x = (x_1, \dots, x_n)\}$ , is studied.

Denote by  $K[\partial/\partial x] = K[\partial/\partial x_1, \dots, \partial/\partial x_n]$  the ring of all symbolic differential polynomials over the field  $K$  of complex numbers. The mapping

$$p(\partial/\partial x) = \sum a_\lambda \frac{\partial^{\lambda_1 + \dots + \lambda_n}}{\partial x_1^{\lambda_1} \dots \partial x_n^{\lambda_n}} \rightarrow p(y) = \sum a_\lambda y_1^{\lambda_1} \dots y_n^{\lambda_n} \quad (2)$$

transforms this ring isomorphically onto the ring  $K[y] = K[y_1, \dots, y_n]$  of all polynomials in the variables  $y_1, \dots, y_n$  with complex coefficients.

The forms  $l_j(\partial/\partial x)$  generate in  $K[\partial/\partial x]$  a certain ideal  $\hat{H}$ . Under the isomorphism (2), the ideal  $\hat{H}$  in  $K[\partial/\partial x]$  corresponds to the ideal  $H$ , generated by the forms  $l_j(y)$ .

The study of the structure of the solutions of system (1) is carried out under the assumption that for the ideal  $H$  its decomposition into an intersection of maximal primary components is known. The results obtained are local in character—they are formulated for some neighborhood of each interior point of the domain  $D$ .

As the “space of values” of the variables  $y_1, \dots, y_n$  take the complex projective space  $\Pi_{n-1} = \{z = (z_1, \dots, z_n)\}$ , where  $z_1^2 + \dots + z_n^2 \neq 0$ . We shall regard  $\Pi_{n-1}$

as the affine space  $A_{n-1} = \{z' = (z_1, \dots, z_{n-1})\}$ , supplemented by the improper hyperplane  $z_n = 0$ .

To an arbitrary ideal  $I'$  of  $K[y'] = K[y_1, \dots, y_{n-1}]$  there corresponds the ideal  $I$  of  $K[y]$ , generated by forms which for  $y_n = 1$  pass into polynomials of  $I'$ . Such an ideal  $I$  is called the homogeneous ideal belonging to  $I'$ . We shall assume that none of the forms  $l_j(y)$  is divisible by  $y_n$ . This can be achieved by a linear transformation of coordinates. In that case, without loss of generality, the ideal  $H$  may be considered homogeneous. It belongs to the ideal  $H'$  of  $K[y']$ , generated by the polynomials  $l_j(y_1, \dots, y_{n-1}, 1)$ ,  $j = 1, \dots, k$ . Suppose

$$H' = [Q'_1, \dots, Q'_r]$$

is a decomposition of  $H'$  in the ring  $K[y']$  into an intersection of maximal primary components  $Q'_\alpha$ . Let  $P'_\alpha$ ,  $\alpha = 1, \dots, r$ , be the prime ideals corresponding to the ideals  $Q'_\alpha$ ;  $\rho_\alpha$  the exponents of the ideals  $P'_\alpha$ ;  $\vec{\xi}_\alpha = (\xi_{\alpha 1}, \dots, \xi_{\alpha n-1})$  the common roots of the ideals  $P'_\alpha$ . Denoting by  $K(\vec{\xi}_\alpha)$  the field of algebraic functions obtained by adjoining to the field  $K$  the elements  $\xi_{\alpha 1}, \dots, \xi_{\alpha n-1}$ , consider in the ring  $K(\vec{\xi}_\alpha)[y]$  the ideal

$$\mathfrak{h}_\alpha = (l_1(y), \dots, l_k(y), P_{\vec{\xi}_\alpha}^{\rho_\alpha}),$$

generated by the forms  $l_j(y)$  and the ideal  $P_{\vec{\xi}_\alpha}^{\rho_\alpha}$ , where  $P_{\vec{\xi}_\alpha} -$

an ideal having as a basis the forms  $y_1 - \xi_{\alpha 1} y_n, \dots, y_{n-1} - \xi_{\alpha n-1} y_n$ . With the help of Hilbert's criterion <sup>(1)</sup> the following lemma is easily proved:

**Lemma 1.** *In order that a polynomial of the ring  $K[y]$  belong to the ideal  $H$ , it is necessary and sufficient that it belong to each ideal  $\mathfrak{H}_\alpha$ ,  $\alpha = 1, \dots, r$ .*

Extend the isomorphism  $K[y] \simeq K[\partial/\partial x]$  to the isomorphism

$$K(\vec{\xi}_\alpha)[y] \simeq K(\vec{\xi}_\alpha)[\partial/\partial x].$$

If under this correspondence  $\mathfrak{H}_\alpha \leftrightarrow \hat{\mathfrak{H}}_\alpha$ , then Lemma 1 implies

**Lemma 2.** *In order that a form of the ring  $K[\partial/\partial x]$  belong to the ideal  $\hat{H}$ , it is necessary and sufficient that it belong to each ideal  $\hat{\mathfrak{H}}_\alpha$ .*

Let  $Q_\alpha$  be the homogeneous ideal in  $K[y]$  belonging to  $Q'_\alpha$ ; let  $\mathfrak{M}_\alpha$  be the variety of roots of the ideal  $Q_\alpha$ . If  $\mathfrak{M}$  is the variety of roots of the ideal  $H$ , then, as is known,

$$\mathfrak{M} = \mathfrak{M}_1 + \dots + \mathfrak{M}_r.$$

We shall call a set  $\mathfrak{N} \subset \mathfrak{M}$  a **defining subset** determining the variety  $\mathfrak{M}$  if, for any polynomial  $P$ , from  $P(y) \equiv 0$  on  $\mathfrak{N}$  it follows that  $P(y) \equiv 0$  on  $\mathfrak{M}$ . Clearly, a defining subset can always be chosen to be no more than countable.

**Lemma 3.** The equality

$$\hat{H} = \bigcap_{\alpha=1}^r \bigcap_{z \in \mathfrak{M}_\alpha} (l_1(\partial/\partial x), \dots, l_k(\partial/\partial x), \hat{P}_z^{\rho_\alpha})$$

holds, where the expression in parentheses denotes the ideal in  $K[\partial/\partial x]$  generated by the forms  $l_j(\partial/\partial x)$  and by the ideal  $\hat{P}_z^{\rho_\alpha}$ , with  $\hat{P}_z$  the ideal having as a basis the forms

$$z_n \partial/\partial x_1 - z_1 \partial/\partial x_n, \dots, z_n \partial/\partial x_{n-1} - z_{n-1} \partial/\partial x_n;$$

$\mathfrak{M}_\alpha$  is a subset determining the variety  $\mathfrak{M}_\alpha$ .

Let  $z$  be an arbitrary point of  $\mathfrak{M}_\alpha$ . We shall call the system of equations

$$l_j(\partial/\partial x)u = 0, \quad j = 1, \dots, k;$$

$$(z_n \partial/\partial x_1 - z_1 \partial/\partial x_n)^{\beta_1} \dots (z_n \partial/\partial x_{n-1} - z_{n-1} \partial/\partial x_n)^{\beta_{n-1}} u = 0 \quad (3)$$

for all integer  $\beta_j \geq 0$ ,  $\beta_1 + \dots + \beta_{n-1} = \rho_\alpha$ , an **elementary system** belonging to the chosen point  $z$  of the variety  $\mathfrak{M}_\alpha$  and to the exponent  $\rho_\alpha$ ,  $\alpha = 1, \dots, r$ .

**Theorem 1.** Let  $u = u(x)$  be an arbitrary solution, analytic in a domain  $D$ , of system (1), and let  $x_0 \in D$ . If  $\{z_l\}$ ,  $l = 1, \dots$ , is a defining subset of the variety  $\mathfrak{M}$ , then, for any  $l$ , for each point  $z_\nu$ ,  $\nu = 1, \dots, l$ , of this sequence one can choose a solution  $u_{l z_\nu}(x, x_0)$  of the corresponding elementary system (3) such that in some neighborhood of the point  $x_0$  the sequence of functions

$$u_l(x, x_0) = \sum_{\nu=1}^l u_{l z_\nu}(x, x_0), \quad l = 1, \dots, \quad (4)$$

converges uniformly to  $u(x)$ .

Having now determined the structure of the solutions of each elementary system, we shall obtain a construction of any analytic solution of system (1); such a solution locally belongs to the closure of the sums (4).

By induction on  $\rho_\alpha$  it is easy to show that every sufficiently smooth solution  $u_z(x)$  of system (3) can be put in the form

$$u_z(x) = \sum_{m=0}^{\rho_\alpha-1} \sum_{|\beta|=m} x'^\beta \varphi_\beta(zx), \quad (5)$$

where

$$x'^\beta = x_1^{\beta_1} \dots x_{n-1}^{\beta_{n-1}}, \quad |\beta| = \beta_1 + \dots + \beta_{n-1}, \quad zx = z_1 x_1 + \dots + z_n x_n.$$

The functions  $\varphi_\beta$ , generally speaking, are not arbitrary, but satisfy relations that are found from the condition that the sum (5) must be a solution of system (1). For the proof of the theorem it is sufficient, as ...

$u_z(x)$  to be homogeneous polynomials. Indeed, without restricting generality, one may assume  $x_0 = 0$ . In a neighborhood of zero every analytic solution of system (1) expands into a series in homogeneous polynomials, each of which is a solution of system (1). Therefore it suffices to consider only those solutions of system (1) that are forms, and to try to expand them (having in mind equality (4)) also into forms.

Replace in the equations of system (3) the coordinates  $z_1, \dots, z_n$  respectively by the components  $\eta_{\alpha 1} = \lambda \xi_{\alpha 1}, \dots, \eta_{\alpha n-1} = \lambda \xi_{\alpha n-1}, \eta_{\alpha n} = \lambda$ , where  $\lambda \neq 0$  is an arbitrary complex number. The resulting system may be called the **elementary system belonging to the common root  $\vec{\eta}_\alpha$  and to the exponent  $\rho_\alpha$** . For such a system in the ring  $K(\vec{\xi}_\alpha)[x]$  we shall consider polynomial solutions, defining the differentiation operations formally. If  $u_{\vec{\eta}_\alpha}(x)$  is a solution of the constructed system which is a form of degree  $q$ , then equality (5) takes the form

$$u_{\vec{\eta}_\alpha}(x) = \sum_{\substack{m=0 \\ m \leq q}}^{\rho_\alpha - 1} \sum_{|\beta|=m} a_\beta x^\beta (\eta_\alpha x)^{q-|\beta|}, \quad (5')$$

where the  $a_\beta$  are constants from the field  $K(\vec{\xi}_\alpha)$ , satisfying the system of algebraic equations:

$$\sum_{\substack{m=|\nu| \\ m \leq q}}^{\rho_\alpha - 1} (q-m)! \sum_{\substack{|\beta|=m \\ \beta \geq \nu}} G_\beta^\nu \frac{\partial^{|\beta|-|\nu|} l_j(y)}{\partial y^{\beta-\nu}} \Big|_{y=\vec{\eta}_\alpha} a_\beta = 0 \quad (6)$$

for all nonnegative integers  $\nu_1, \dots, \nu_{n-1}$  for which  $|\nu| \leq \rho_\alpha - 1, j = 1, \dots, k$ . Here  $G_\beta^\nu = C_{\beta_1}^{\nu_1} \dots C_{\beta_{n-1}}^{\nu_{n-1}}, \partial y^{\beta-\nu} = \partial y_1^{\beta_1-\nu_1} \dots \partial y_{n-1}^{\beta_{n-1}-\nu_{n-1}}, \beta_j \geq \nu_j (\beta \geq \nu)$ .

Let  $r_\alpha$  be the rank of system (6), and  $n_\alpha$  the number of all undetermined coefficients in the sum of the form (5'). Denote by  $\Delta_\alpha$  one of the determinants of order  $r_\alpha$  of the system, different from zero. Let  $a_{\beta_1}, \dots, a_{\beta_{r_\alpha}}$  be the unknown coefficients that enter into the columns of the matrix  $\Delta_\alpha$ . They will be linear combinations with coefficients from the field  $K(\vec{\xi}_\alpha)$  of the remaining  $n_\alpha - r_\alpha$  unknowns  $a_\beta$ , and the latter may be regarded as arbitrary constants. Having found the indicated linear combinations and substituting the results in (5'), we obtain that  $u_{\vec{\eta}_\alpha}(x)$  is a linear combination of forms of the form

$$p_{\delta \vec{\eta}_\alpha}(x) = x^{\gamma_\delta} (\eta_\alpha x)^{q-|\gamma_\delta|} + \sum_{j=1}^{r_\alpha} c_{\beta_j \delta}(\vec{\eta}_\alpha) x^{\beta_j} (\eta_\alpha x)^{q-|\beta_j|}, \quad \delta = 1, \dots, n_\alpha - r_\alpha,$$

where  $c_{\beta_j \delta}(\vec{\eta}_\alpha)$  are constants from  $K(\vec{\xi}_\alpha)$ ,  $|\gamma_\delta| = \gamma_{\delta 1} + \dots + \gamma_{\delta n-1} \leq \rho_\alpha - 1, \gamma_\delta \neq \beta_j, j = 1, \dots, r_\alpha$ . The polynomials  $p_{\delta \vec{\eta}_\alpha}$  are, evidently, also solutions of

the elementary system belonging to the common root  $\vec{\eta}_\alpha$  and the exponent  $\rho_\alpha$ . Hence the structure of a solution of system (3) which is an arbitrary form of degree  $q$ , for those points  $z \in \mathfrak{M}_\alpha$  for which  $\Delta_\alpha \neq 0$ , is obtained immediately:

$$p_{\delta z}(x) = x^{\gamma_\delta}(zx)^{q-|\gamma_\delta|} + \sum_{j=1}^{r_\alpha} c_{\beta_j \delta}(z)x^{\beta_j}(zx)^{q-|\beta_j|}, \quad \delta = 1, \dots, n_\alpha - r_\alpha.$$

Let  $t_q(x)$  be a homogeneous polynomial from  $K[x]$  satisfying system (1). The proof of the theorem now reduces to proving the existence of points  $z_1, \dots, z_\omega$ , belonging to  $\{z\}$ , and corresponding to these

points of the polynomials  $p_{\delta_j z_j}(\mathbf{x})$ , and also constants  $\lambda_j$  such that

$$t_q(\mathbf{x}) = \sum_1^\omega \lambda_j p_{\delta_j z_j}(\mathbf{x}). \quad (7)$$

Putting  $\omega = C_{q+n-1}^{n-1}$  and equating the coefficients of like products  $\mathbf{x}^\alpha$  on the right and on the left in equality (7), we arrive at a system of equations with respect to  $\lambda_j$ , which turns out to be consistent for some set of points  $z_j$  of the sequence  $\{z_l\}$  and specially chosen indices  $\delta_j$ . Without giving the system itself, we note that the proof of its consistency reduces, in essence, to the proof of the following fact.

**Lemma 4.** If a form  $p(\mathbf{y})$  of degree  $q$ ,  $p(\mathbf{y}) \in K[\mathbf{y}]$ , satisfies the system of equalities

$$(q - |\gamma_\delta|)! \frac{\partial^{|\gamma_\delta|} p(\mathbf{y})}{\partial y_1^{\gamma_{\delta_1}} \dots \partial y_{n-1}^{\gamma_{\delta_{n-1}}}} \Big|_{\mathbf{y}=\vec{\eta}_\alpha} + \sum_{j=1}^{r_\alpha} (q - |\beta_j|)! c_{\beta_j \delta}(\vec{\eta}_\alpha) \times \\ \times \frac{\partial^{|\beta_j|} p(\mathbf{y})}{\partial y_1^{\beta_{j1}} \dots \partial y_{n-1}^{\beta_{jn-1}}} \Big|_{\mathbf{y}=\vec{\eta}_\alpha} = 0, \quad \delta = 1, \dots, n_\alpha - r_\alpha,$$

then  $p(\mathbf{y}) \in \mathfrak{h}_\alpha$ .

Thus, Theorem 1 is valid. At the same time, information has been obtained on the local structure of the analytic solutions of system (1).

Suppose now that the space  $R_n$  is real. Let the coefficients of the forms  $l_j$  also be real. We shall consider only real solutions of system (1). For such systems and solutions, certain results in the direction indicated by the present paper were obtained earlier. For example, in (2) the case  $H = (0)$ ,  $\mathfrak{M} = \Pi_{n-1}$ , was studied. In (3) the ideal  $H$  was assumed to be zero-dimensional, and its variety consisted of simple real points.

If  $z$  is a real point of  $\mathfrak{M}_\alpha$ , then every real solution of system (3) is represented in the form (5), where  $\varphi_\beta$  are real functions. If, however,  $z$  is an imaginary point of  $\mathfrak{M}_\alpha$ , then, separating in (5) the real part, we obtain ( $z = \mathbf{a} + i\mathbf{b}$ ;  $\mathbf{a}$  and  $\mathbf{b}$  are real vectors):

$$u_z(\mathbf{x}) = \sum_{m=0}^{\rho_\alpha-1} \sum_{|\beta|=m} \mathbf{x}'^\beta \psi_\beta(\mathbf{ax}, \mathbf{bx}), \quad (8)$$

where  $\psi_\beta(\mathbf{ax}, \mathbf{bx}) = \operatorname{Re} \varphi_\beta(\mathbf{zx})$ . The functions  $\psi_\beta(v, w)$  will be harmonic with respect to  $v$  and  $w$ . Consequently, the real solutions of the elementary system belonging to an imaginary point  $z$  of the variety  $\mathfrak{M}_\alpha$  are constructed with the aid of harmonic functions, starting from the sum (8).

In the case under consideration, the polynomials  $p_{\delta z}(\mathbf{x})$ ,  $\delta = 1, \dots, n_\alpha - r_\alpha$ , constructed for a real point  $z \in \mathfrak{M}_\alpha$ , are real. To each of the polynomials  $p_\delta(\mathbf{x})$  constructed for an imaginary point  $z \in \mathfrak{M}$ , there corresponds a pair of linearly independent real polynomials  $\operatorname{Re} p_{\delta z}(\mathbf{x})$  and  $\operatorname{Im} p_{\delta z}(\mathbf{x})$ . These polynomials are formed from polynomials harmonic with respect to  $v$  and  $w$ ,  $\operatorname{Re}(v + iw)^{q-|\beta_j|}$  and  $\operatorname{Im}(v + iw)^{q-|\beta_j|}$ ,  $v = \mathbf{ax}$ ,  $w = \mathbf{bx}$ . It follows that, in order to construct all possible analytic solutions of arbitrary real systems (1), in addition to the set of powers  $v^\nu$ ,  $\nu = 1, \dots$  ( $v = \mathbf{zx}$ ,  $z$  a real point of  $\mathfrak{M}$ ), one also needs the set of harmonic polynomials  $\operatorname{Re}(v + iw)^\nu$  and  $\operatorname{Im}(v + iw)^\nu$ ,  $\nu = 1, \dots$  ( $v = \mathbf{ax}$ ,  $w = \mathbf{bx}$ ,  $z = \mathbf{a} + i\mathbf{b}$  an imaginary point of  $\mathfrak{M}$ ). In particular, if system (1) is elliptic, then an arbitrary solution of this system is constructed with the aid of harmonic polynomials in two variables.

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*Note: Figure translations are in progress. See original paper for figures.*

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