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# MATHEMATICS

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1965

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**Abstract**

**Full Text**

MATHEMATICS

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## THE HOMOGENEOUS RIEMANN PROBLEM ON CLOSED RIEMANN SURFACES

*(Presented by Academician I. N. Vekua, August 3, 1964)*

Let, on a closed Riemann surface  $R$  of genus  $\rho$ , there be given a contour  $\Gamma$ , consisting of a finite number of mutually nonintersecting smooth closed oriented curves, and a function  $G \in H(\Gamma)$ ,  $G \neq 0$ . In [1] it was established that, for  $\chi = \text{ind}_{\Gamma} G \geq 0$ , the solvability of the homogeneous Riemann problem for analytic functions, defined by the boundary condition on the contour  $\Gamma$

$$\Phi^+ = G\Phi^-, \quad (1)$$

is equivalent to the solvability of the congruence

$$\sum_{j=1}^{\chi} u_{\nu}(D_j) \equiv \lambda_{\nu} \pmod{\text{periods}} \quad (\nu = 1, 2, \dots, \rho), \quad (2)$$

where  $\lambda_1, \lambda_2, \dots, \lambda_{\rho}$  are numbers depending on the coefficient  $G$  and on the surface  $R$  (for all notation see [1, 2]), and, in terms of the Riemann  $\vartheta$ -function, necessary and sufficient conditions for the solvability of this congruence were given. In the present work we obtain explicit expressions for the number  $K$  of linearly independent solutions of problem (1) for  $0 < \chi \leq \rho - 1$  (Theorem 1) and for  $\rho \leq \chi \leq 2\rho - 2$  (Theorem 2).

Define, on the  $\rho$ -dimensional complex space, the function  $\chi(z_1, z_2, \dots, z_{\rho}) = \chi(z_{\nu})$  as follows:  $\chi(z_{\nu}^0) = -1$  if  $\vartheta(z_{\nu}^0) = \vartheta(z_1^0, z_2^0, \dots, z_{\rho}^0) \neq 0$ , and  $\chi(z_{\nu}^0) = n$  if the function  $\vartheta(z_{\nu})$ , together with all its partial derivatives up to order  $n$  inclusive, vanishes at the point  $(z_1^0, z_2^0, \dots, z_{\rho}^0)$ , but at least one derivative of the function  $\vartheta(z_{\nu})$  of order  $n + 1$  at this point does not vanish. From the properties of the Riemann  $\vartheta$ -function it follows that, if  $(e_{\nu}) \equiv (g_{\nu}) \pmod{\text{periods}}$ , then  $\chi(e_{\nu}) = \chi(g_{\nu})$  [3].

The subsequent arguments are based on the following assertion [3]: if  $\chi(e_{\nu} + k_{\nu}) = n \geq 0$ , then

$$e_\nu = \sum_{j=1}^{\rho-1} u_\nu(D_j) \pmod{\text{periods}} \quad (\nu = 1, 2, \dots, \rho), \quad (3)$$

and, in modern terms,  $\dim D_1 D_2 \dots D_{\rho-1} = n + 1$ , and conversely,  $(k_1, k_2, \dots, k_\rho)$  are fixed numbers depending only on the surface; [2], Ch. IX, equality (16)).

**Lemma 1.** *Any integral divisor of order  $\chi$ ,  $0 < \chi \leq \rho - 1$ ,  $D_1 D_2 \dots D_\chi$ , can be extended to a divisor of order  $\rho - 1$ ,  $D_1 D_2 \dots D_{\rho-1}$ , in such a way that*

$$\dim D_1 D_2 \dots D_\chi = \dim D_1 D_2 \dots D_\chi D_{\chi+1} \dots D_{\rho-1}.$$

**Theorem 1.** *If  $0 < \chi \leq \rho - 1$ , then*

$$K = \min_{P_j \in R} \chi \left( \sum_{j=1}^{\rho-\chi-1} u_\nu(P_j) + \lambda_\nu + k_\nu \right) + 1. \quad (4)$$

**Proof.** Consider first the case  $K = 0$ . By virtue of Theorem 2 of [1], there exists a divisor  $P'_1 P'_2 \dots P'_{\rho-\chi-1}$  such that

$$\vartheta \left( \sum_{j=1}^{\rho-\chi-1} u_\nu(P'_j) + \lambda_\nu + k_\nu \right) \neq 0, \quad (5)$$

and therefore

$$\min_{P_j \in R} \chi \left( \sum_{j=1}^{\rho-\chi-1} u_\nu(P_j) + \lambda_\nu + k_\nu \right) = -1. \quad (6)$$

Conversely, from (6), by definition, (5) follows; whence, on the basis of Theorem 2 of [1], we have  $K = 0$ .

If  $K \geq 1$ , then comparison (2) is solvable and  $K = \dim D_1 D_2 \dots D_\chi$ . Using Lemma 1, we obtain

$$\begin{aligned} K &= \dim D_1 D_2 \dots D_\chi = \min_{P_j \in R} \dim D_1 D_2 \dots D_\chi P_1 P_2 \dots P_{\rho-\chi-1} = \\ &= \min_{P_j \in R} \chi \left( \sum_{j=1}^{\chi} u_\nu(D_j) + \sum_{j=1}^{\rho-\chi-1} u_\nu(P_j) + k_\nu \right) + 1 = \\ &= \min_{P_j \in R} \chi \left( \sum_{j=1}^{\rho-\chi-1} u_\nu(P_j) + \lambda_\nu + k_\nu \right) + 1. \end{aligned}$$

The theorem is proved.

**Lemma 2.** *If  $\dim P_1 P_2 \dots P_\rho > 1$ , then, in order to satisfy the comparison*

$$\sum_{j=1}^{\rho} u_\nu(Q_j) \equiv \sum_{j=1}^{\rho} u_\nu(P_j) \pmod{\text{periods}} \quad (\nu = 1, 2, \dots, \rho),$$

*at least one of the points  $Q_1, Q_2, \dots, Q_\rho$  may be prescribed arbitrarily, and conversely.*

Let now  $\chi \geq \rho$ . In this case comparison (2) is always solvable, and, moreover,  $\chi - \rho$  points  $D_{\rho+1}, \dots, D_\chi$  may be prescribed arbitrarily.

**Theorem 2.** *If  $\rho \leq \chi \leq 2\rho - 2$ , then*

$$K = \min_{P_j \in R} \chi \left( \lambda_\nu - \sum_{j=1}^{\chi-\rho+1} u_\nu(P_j) + k_\nu \right) + \chi - \rho + 2. \quad (7)$$

**Proof.** From the Riemann-Roch theorem it follows that  $K \geq \chi - \rho + 1$ . Consider first the case  $K = \chi - \rho + 1$ . It is known ((2), lemma on p. 154) that, in the equivalence class of divisors satisfying comparison (2), one can find such a divisor  $D'_1 D'_2 \dots D'_\rho D'_{\rho+1} \dots D'_\chi$  that  $\dim D'_1 D'_2 \dots D'_\rho = 1$ . Then the function

$$\vartheta \left( u(P) - \left( \sum_{j=1}^{\rho} u_\nu(D'_j) + k_\nu \right) \right)$$

is not identically equal to zero, and its zeros are the points  $D'_1, D'_2, \dots, D'_\rho$  [2]. Taking a point  $D'$  different from  $D'_1, \dots, D'_\rho$ , we shall have

$$\begin{aligned} \vartheta \left( \lambda_\nu - \sum_{j=\rho+1}^{\chi} u_\nu(D'_j) - u_\nu(D') + k_\nu \right) &= -\vartheta \left( u_\nu(D') - \right. \\ &\left. - \left( \lambda_\nu - \sum_{j=\rho+1}^{\chi} u_\nu(D'_j) + k_\nu \right) \right) = C \vartheta \left( u_\nu(D') - \left( \sum_{j=1}^{\rho} u_\nu(D'_j) + k_\nu \right) \right) \neq 0 \end{aligned}$$

( $C$  is a nonzero constant), whence it follows that

$$\min_{P_j \in R} \chi \left( \lambda_\nu - \sum_{j=1}^{\chi-\rho+1} u_\nu(P_j) + k_\nu \right) = -1. \quad (8)$$

Conversely, suppose that (8) holds, or, what is the same thing, there exists a divisor  $Q'_1 Q'_2 \dots Q'_{\chi-\rho+1}$  such that

$$\vartheta \left( \lambda_\nu - \sum_{j=1}^{\chi-\rho+1} u_\nu(Q'_j) + k_\nu \right) \neq 0. \quad (9)$$

Inequality (9) means that the congruence

$$\sum_{j=1}^{\rho-1} u_\nu(D_j) \equiv \lambda_\nu - \sum_{j=1}^{\chi-\rho+1} u_\nu(Q'_j) \pmod{\text{periods}} \quad (\nu = 1, 2, \dots, \rho) \quad (10)$$

is unsolvable; hence, on the basis of Lemma 2, we conclude that if the divisor  $D_1 D_2 \cdots D_\rho$  is a solution of the congruence

$$\sum_{j=1}^{\rho} u_\nu(D_j) \equiv \lambda_\nu - \sum_{j=1}^{\chi-\rho} u_\nu(Q'_j) \pmod{\text{periods}} \quad (\nu = 1, 2, \dots, \rho), \quad (11)$$

then  $\dim D_1 D_2 \cdots D_\rho = 1$ ; consequently,

$$\dim D_1 D_2 \cdots D_\rho Q'_1 Q'_2 \cdots Q'_{\rho-\chi} \leq \chi - \rho + 1.$$

On the other hand, by the Riemann-Roch theorem,

$$\dim D_1 D_2 \cdots D_\rho Q'_1 Q'_2 \cdots Q'_{\rho-\chi} \geq \chi - \rho + 1.$$

Finally we have

$$K = \dim D_1 D_2 \cdots D_\rho Q'_1 Q'_2 \cdots Q'_{\rho-\chi} = \chi - \rho + 1.$$

Let now  $K > \chi - \rho + 1$ . From Lemma 2 it follows that congruence (10) is solvable with respect to the points  $D_1, D_2, \dots, D_{\rho-1}$ , whatever the points  $Q'_1, Q'_2, \dots, Q'_{\chi-\rho+1}$  may be. Denote by  $\Delta$  the equivalence class of divisors satisfying congruence (2), and by  $\delta$  the set of all divisors of order  $\rho - 1$  which are sub-divisors of divisors of the class  $\Delta$ . From the above-mentioned lemma ((2), p. 154) it follows that

$$\min_{D \in \delta} \dim D = \dim \Delta - \chi + \rho + 1 = K - \chi + \rho - 1.$$

As is easily seen, the divisors  $D_1 D_2 \cdots D_{\rho-1} = D$  which are solutions of congruence (10) exhaust the whole set  $\delta$ , when the points  $Q'_1, Q'_2, \dots, Q'_{\chi-\rho+1}$  independently describe the entire surface  $R$ . Thus,

$$\begin{aligned}
 K &= \chi - \rho + 1 + \min_{D \in \delta} \dim D \\
 &= \min_{D \in \delta} \chi \left( \sum_{j=1}^{\rho-1} u_\nu(D_j) + k_\nu \right) + \chi - \rho + 2 \\
 &= \min_{P_j \in R} \chi \left( \lambda_\nu - \sum_{j=1}^{\chi-\rho+1} u_\nu(P_j) + k_\nu \right) + \chi - \rho + 2.
 \end{aligned}$$

The theorem is proved. Formula (7) can be obtained from the relation

$$K = K' + \chi - \rho + 1,$$

where  $K'$  is the number of linearly independent solutions of the problem conjugate to the original one. The conjugate problem is equivalent to a certain congruence with  $2\rho - 2 - \chi$  unknown points, and, therefore, Theorem 1 may be used to calculate the number  $K'$ . Finally, we note that for  $\chi > 2\rho - 2$ ,

$$K = \chi - \rho + 1.$$

The theorems stated above are applicable to the study of the Hilbert problem determined by prescribing on the boundary of a finite surface (in particular, of a multiply connected domain) the condition

$$\operatorname{Re}[(a - ib)F] = 0,$$

since this problem, in a known way (4), is reduced to problem (1) on the double of the given Riemann surface.

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Received  
17 VI 1964

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*Note: Figure translations are in progress. See original paper for figures.*

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