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**Abstract**

**Full Text**

E. K. Isakova

## ON THE PRINCIPLE OF LIMITING AMPLITUDE FOR SOME BOUNDARY-VALUE PROBLEMS IN THE PLANE

(Presented by Academician A. A. Dorodnitsyn, 23 XII 1964)

Let us consider the equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} + P(\partial/\partial x)u(x, t) = f(x)e^{i\omega t}, \quad x > 0; t > 0, \quad (1)$$

where  $P(\alpha) = a_{2m}\alpha^{2m} + \dots + a_1\alpha + a_0$  is a polynomial of degree  $2m$  with constant real coefficients,  $\text{sign } a_{2m} = (-1)^m$ ;  $f(x)$  is a finite function for  $x \geq 0$ ;  $\omega > 0$  is a real number.

Let  $u(x, t)$  be a solution of equation (1), bounded in the domain  $D\{x > 0, t > 0\}$ , satisfying on the boundary of this domain the conditions

$$u|_{t=0} = 0, \quad u_t|_{t=0} = \varphi(x), \quad x \geq 0; \quad (2)$$

$$\sum_{j=0}^{\sigma_i} b_{ij} \frac{\partial^j u}{\partial x^j} \Big|_{x=0} = 0, \quad 2m-1 \geq \sigma_1 > \dots > \sigma_m \geq 0, \quad (3)$$

where  $\varphi(x)$  is a finite function;  $b_{ij}$ ,  $j = 0, \dots, \sigma_i$ ,  $i = 1, \dots, m$ , are real constants,  $b_{i\sigma_i} \neq 0$ ,  $i = 1, \dots, m$ . We are interested in the behavior of the solution  $u(x, t)$  of problem (1), (2), (3) as  $t \rightarrow \infty$ .

Consider the ordinary differential equation

$$P(d/dx)w(x, \lambda) + \lambda^2 w(x, \lambda) = F(x), \quad x > 0, \quad (4)$$

with complex parameter  $\lambda$  and with a finite function  $F(x)$ .

Denote by  $\alpha_1(\lambda), \dots, \alpha_{2m}(\lambda)$  the  $\alpha$ -roots of the polynomial  $P(\alpha) + \lambda^2$ . Then, for sufficiently large  $\text{Re } \lambda > 0$ , one half of these roots have positive real part:  $\text{Re } \alpha_j(\lambda) > 0$ ,  $j = 1, \dots, m$ , and the other half have negative real part:  $\text{Re } \alpha_{m+j}(\lambda) < 0$ ,  $j = 1, \dots, m$ . Denote by  $\lambda = \lambda_j$ ,  $j = 1, \dots, N$ , all finite branch points of the roots  $\alpha_1(\lambda), \dots, \alpha_{2m}(\lambda)$ . The order  $s_j$  of the branch point  $\lambda = \lambda_j$ ,

$j = 1, \dots, N$ , will mean the greatest number of roots which at the point  $\lambda = \lambda_j$  take one and the same value.

**Lemma 1.** For sufficiently large  $\operatorname{Re} \lambda > 0$  there exists a unique solution  $w(x, \lambda)$ , bounded for  $0 \leq x < \infty$ , of problem (4)–(3).

There exists a constant  $A_1 = A_1(b)$  such that for  $|\lambda| > b > 0$

$$|w(x, \lambda)| \leq A_1/|\lambda|^{(2m-1)/m} \quad \text{for all } x \in (0, \infty).$$

The function  $w(x, \lambda)$  is an analytic function in the  $\lambda$ -plane and may have critical points (branch points) only at  $\lambda = \infty$ ,  $\lambda = \lambda_j$ ,  $j = 1, \dots, N$ . Cut the complex  $\lambda$ -plane along straight lines parallel to the real axis, to the left of the points  $\lambda = \lambda_j$ ,  $j = 1, \dots, N$ , to infinity. Everywhere in what follows we shall consider only that branch of the function  $w(x, \lambda)$  which is the analytic continuation of the function  $w(x, \lambda)$  defined by Lemma 1.

**Lemma 2.** In a neighborhood of the point  $\lambda = \lambda_j$ ,  $j = 1, \dots, N$ , the function  $w(x, \lambda)$  has the representation

$$w(x, \lambda) = A_2(x)/(\lambda - \lambda_j)^{1-2/s_j} + \tilde{w}_j(x, \lambda), \quad j = 1, \dots, N,$$

where  $A_2(x)$  is uniformly bounded for  $x \in [0, X]$ ;  $\tilde{w}_j(x, \lambda) =$

$$= o(1/|\lambda - \lambda_j|^{1-2/s_j})$$

uniformly in  $x$  for  $x \in [0, X]$ ,  $X > 0$  an arbitrary fixed number.

With the aid of Lemmas 1 and 2, Theorems 1 and 2 are proved.

**Theorem 1.** Let the function  $f(x)$  be continuously differentiable, and let the function  $\varphi(x)$  have  $m + 1$  continuous derivatives. Let  $f(0) = 0$ ,  $\varphi(0) = 0$ ,  $\varphi'(0) = 0, \dots, \varphi^{(m)}(0) = 0$ .

Then there exists a unique solution  $u(x, t)$  of problem (1), (2), (3) in the domain  $\{x > 0, t > 0\}$ , bounded in  $x$ ,  $x \in (0, \infty)$ , for any  $t > 0$ , together with all derivatives with respect to  $x$  up to order  $(2m - 1)$ .

**Theorem 2.** If  $\operatorname{Re} \lambda_j = 0$ ,  $j = 1, \dots, N$ ,  $i\omega \neq \lambda_j$ ,  $j = 1, \dots, N$ , then there exists a function  $v(x)$  such that, as  $t \rightarrow \infty$ ,

$$|u(x, t)e^{-i\omega t} - v(x)| \leq A_3(x)t^{-2/s}, \quad (5)$$

where  $A_3(x)$  is uniformly bounded in  $x \in [0, X]$ ;  $X > 0$  is an arbitrary fixed number;  $s$  is the largest of the numbers  $s_1, \dots, s_N$ .

If the order of equation (1)  $m > 1$ , then the exponent  $-2/s$  in inequality (5) is exact; if  $m = 1$ , then inequality (5) may be replaced by the inequality

$$|u(x, t)e^{-i\omega t} - v(x)| \leq A_3(x)e^{-\beta t},$$

where  $\beta$  is an arbitrary positive number.

The function  $v(x)$  in Theorem 2 is called the limiting amplitude. It is the limit, as  $\lambda \rightarrow i\omega$ , of the function  $w(x, \lambda)$  defined in Lemma 1, if in equation (4) one sets  $F(x) \equiv f(x)$ . For some special cases of equation (1), the function  $v(x)$  can be determined without using analytic continuation of the function  $w(x, \lambda)$ .

**Theorem 3.** If the polynomial  $P(\alpha)$  is even, then the function  $v(x)$  is the unique solution of the equation

$$P(d/dx)v(x) - \omega^2v(x) = f(x), \quad x > 0,$$

which for  $x = 0$  satisfies conditions (3), and as  $x \rightarrow \infty$  satisfies the conditions

$$|v(x)| < C, \quad C > 0 \text{ is a constant,}$$

$$dP_j(d/dx)v(x)/dx + i\theta_{jP}j(d/dx)v(x) = o(1), \quad j = 1, \dots, n, \quad (6)$$

where

$$P_j(\alpha) = (P(\alpha) - \omega^2)/(\alpha^2 + \theta_j^2), \quad j = 1, \dots, n;$$

$$P(\alpha) = (\alpha^2 + \theta_1^2) \dots (\alpha^2 + \theta_n^2) \tilde{P}(\alpha);$$

$\theta_j^2$  are real,  $j = 1, \dots, n$ , and the polynomial  $\tilde{P}(\alpha)$  has no imaginary roots.

Conditions (6) are radiation conditions of the Sommerfeld-Vekua type (1). We note that Theorem 3 remains valid also under less stringent restrictions on the polynomial  $P(\alpha)$ .

The problem of the limiting amplitude was first posed in work (2) for the Cauchy problem for the wave equation in three-dimensional space. For exterior boundary-value problems for the wave equation, the limiting-amplitude problem was considered in works (3-5).

The results obtained above remain valid also in the case where, instead of problem (1), (2), (3), one considers problem (1'), (2'), (3), where (1') is an equation of Schrödinger type (it is obtained from (1) by replacing  $\partial^2/\partial t^2$  by  $i\partial/\partial t$ ), and (2') is  $u(x, t)|_{t=0} = 0$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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