GENERALIZED SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract

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GENERALIZED SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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1. In ([1]) generalized solutions of differential equations in Hilbert space were studied. The coercivity inequalities established in ([2]) for equations in a Banach space made it possible to obtain, for such equations, analogous and, in a certain sense, stronger results. Namely, for solutions of homogeneous equations with a variable operator, under minimal restrictions on the smoothness of this operator, it is possible to obtain the same estimates (point singularities) as in the case of a constant operator (Theorems 4 and 5). Such estimates are important in applications to nonlinear equations (see, for example, ([1], [3])).

2. Let A be a strongly positive operator in a Banach space E. This means that A generates an analytic semigroup $\exp\{-tA\}$, whose norm decreases exponentially. From the operator A we construct the spaces $E_\alpha(A)$ $(0 < \alpha < 1)$ with norms $|v|_\alpha^A$. If A_1 is strongly positive and $D(A_1) = D(A)$, then $E_\alpha(A_1) = E_\alpha(A)$ and the norms $|v|_\alpha^{A_1}$ and $|v|_\alpha^A$ are equivalent (see ([2], [4])). Therefore, in what follows, in the notation of spaces and norms constructed from operators with the same domain of definition, the designation of the operator is omitted.

Consider in E the problem

$$v' + Av = f(t) \quad (0 \le t \le T), \qquad v(0) = v_0. \tag{1}$$

By a generalized solution of problem (1) in $B_p([0,T], E)^*$ $(1 < p < \infty)$ we shall mean an absolutely continuous function $v(t)$ on $[0,T]$ which satisfies the equation and the initial condition (1) almost everywhere and has the property:

(a) the functions v', $Av \in B_p$, and the function $v(t)$ is continuous in $E_{1/q}$ $(1/p + 1/q = 1)$.

If for every function $f(t) \in B_p$ and for every element $v_0 \in E_{1/q}$ there exists a unique generalized solution in B_p of problem (1) and the inequality

$$\|v'\|_{B_p} + \|Av\|_{B_p} + \max_{0\le t\le T} |v(t)|_{1/q} \le K_p(A)(\|f\|_{B_p} + |v_0|_{1/p}), \tag{2}$$

holds, then we shall say that coercivity in B_p holds for problem (1). If this fact holds in some one B_{p_0}, then it holds in every B_p ([2]).

Theorem 1. *Let coercivity in B_p hold for problem (1). Then, for the analogous problem with an operator A_1 having the same domain of definition as A, coercivity in B_p also holds.*

* $B_p([0,T],E)$ is the Bochner space with norm $\|v\|_{B_p} = \left(\int_0^T \|v(t)\|^p\,dt\right)^{1/p}$

(see ([5])).

This theorem makes it possible to establish coercivity for equations with a complex operator A_1, if it holds for equations with a simple operator A.

3. Let us consider a problem more general than (1),

$$v' + A(t)v + F(t)v = f(t) \quad (0 \le t \le T), \qquad v(0) = v_0. \tag{3}$$

Suppose that $A(t)$, for each $t \in [0,T]$, is strongly positive, $D[A(t)] = D[A(0)] = D$, and the operator-function $A(t)A^{-1}(0)$ has only discontinuities of the first kind. Suppose that $F(t)$ is closed, $D[F(t)] \supset D$, and the operator-function $F(t)A^{-1}(0)$ is strongly measurable.

By a generalized solution of problem (3) in B_p we shall mean an absolutely continuous function $v(t)$ on $[0,T]$ which, for almost all t, satisfies the equation and the initial condition (4), and possesses property (a) with the operator $A = A(0)$.

Theorem 2. *Suppose that for each fixed $t \in [0,T]$, for problem (1) with the operator $A = A(t)$, coercivity holds in B_p. Suppose that for all $t \in [0,T]$ and $v \in D$ the inequality $\|F(t)v\| \le \delta(t)\|A(t)v\| + C\|v\|$ holds, with $\delta(t)K_p[A(t)] \le \delta < 1$, where $K_p[A(t)]$ is the constant occurring in inequality (2). Then, for any $f(t) \in B_p$ and $v_0 \in E_{1/q}$, problem (3) has a unique generalized solution in B_p, and inequality (2) holds with $A = A(0)$.*

This theorem generalizes Theorem 1 of (1).

4. Consider the homogeneous problem

$$v' + A(t)v = 0 \quad (\tau \le t \le T), \qquad v(\tau) = v_0.$$

We shall denote its solution by $U(t,\tau)v_0$. By Theorem 2, $U(t,\tau)$ is an operator-function, strongly continuous jointly in t and τ, for $0 \le \tau \le t \le T$, in any space E_α, satisfying the condition $U(t,\tau) = U(t,s)U(s,\tau)$ $(\tau \le s \le t)$. The following holds (cf. (1), Theorem 2).

Theorem 3. *For any $0 < \alpha \le \beta < 1$, $0 \le \tau \le t \le T$, the inequalities hold*

$$|[U(t,\tau) - I]v|_\alpha \le C(\alpha,\beta)|t-\tau|^{\beta-\alpha}|v|_\beta \qquad (v \in E_\beta),$$

$$\left[\int_\tau^t |U(s,\tau)v|_\beta^{1/(\beta-\alpha)}\, ds\right]^{\beta-\alpha} \le C(\alpha,\beta)|v|_\alpha \qquad (v \in E_\alpha).$$

If $\alpha < \beta$, then in the left-hand sides of these inequalities the norms $|w|_\gamma$ may be replaced by the norms $\|A^\gamma(0)w\|$.

In proving this theorem, inequalities of the form (2), moment inequalities from (4), and the inequality

$$|v|_\beta \le C(\alpha-\beta,\gamma-\beta)|v|_\alpha^{(\beta-\gamma)/(\alpha-\gamma)}\|A^\gamma(0)v\|^{(\alpha-\beta)/(\alpha-\gamma)}$$

$$(v \in E_\delta,\ \delta = \max\{\alpha,\gamma\}),$$

valid for any $\alpha \in (0,1)$, $\gamma \in [0,1]$, $\alpha \ne \gamma$, and $\beta \in (\alpha,\gamma)$, are used.

The second of the inequalities of Theorem 3 means that the operator $U(t,\tau)$ acts not only in E_α, but, for almost all $t \ge \tau$, from E_α into E_β.

This assertion can be sharpened.

Theorem 4. *For any $0 < \alpha \le \beta < 1$ and $0 \le \tau \le t \le T$, the inequality*

$$|U(t,\tau)v|_\beta \le C(\alpha,\beta)|t-\tau|^{\alpha-\beta}|v|_\alpha \qquad (v \in E_\alpha). \tag{4}$$

holds. If $\alpha < \beta$, the norm $|w|_\beta$ on the left may be replaced by the norm $\|A^\beta(0)w\|$.

Theorem 5. *Suppose that for any $t \in [0,T]$ and $\alpha \in (0,1)$, the operator $A(t)$ admits closure to a bounded operator from E_α to $E^*_{1-\alpha}$, and suppose that this closure is an operator-function having only discontinuities of the first kind. Then, for any $0 < \alpha < 1$ and $0 \le \tau \le t \le T$, the inequality*

$$|U(t,\tau)v|_\alpha \le C(\alpha)|t-\tau|^{-\alpha}\|v\| \qquad (v \in E). \tag{5}$$

holds.

If, on the left, the norm $|w|_\alpha$ is replaced by the norm $\|A^\alpha(0)w\|$, then (5) is valid for $\alpha \geqslant 0$.

We note that the conditions of Theorem 5 are satisfied when $A(t)$ is an elliptic operator in the space $L_p(\Omega)$ with normal boundary conditions (for the definition of a normal elliptic operator see, for example, (6)).

We outline the proofs of Theorems 4 and 5. From the identity

$$v'(t) + A(\tau)v(t) = [A(\tau) - A(t)]v(t) \qquad (\tau \leqslant t \leqslant T),$$

which is satisfied by the function $v(t) = U(t,\tau)v$, it follows that

$$U(t,\tau)v = \exp\{-(t-\tau)A(\tau)\}v +$$

$$+ \int_\tau^t \exp\{-(t-s)A(\tau)\}[A(\tau) - A(s)]U(s,\tau)v\,ds = \exp\{-(t-\tau)A(\tau)\}v +$$

$$+ \int_{(t+\tau)/2}^t \exp\{-(t-s)A(\tau)\}[A(\tau) - A(s)]U\left(s, \frac{t+\tau}{2}\right) ds\, U\left(\frac{t+\tau}{2}, \tau\right) v +$$

$$+ \exp\left\{-\frac{t-\tau}{2}A(\tau)\right\} \int_\tau^{(t+\tau)/2} \exp\left\{-\left(\frac{t+\tau}{2} - s\right) A(\tau)\right\} \times$$

$$\times [A(\tau) - A(s)]U(s,\tau)v\,ds. \tag{6}$$

For simplicity, suppose that $A(t)A^{-1}(0)$ is continuous. Then it follows from (6) that

$$|U(t,\tau)v|_\beta \leqslant C|t-\tau|^{\alpha-\beta}|v|_\alpha + \varepsilon \left| U\left(\frac{t+\tau}{2}, \tau\right) v \right|_\beta + C|t-\tau|^{\alpha-\beta}|v|_\alpha,$$

where $\varepsilon \to 0$ as $t - \tau \to 0$. Here, to estimate the integral

$$\int_\tau^{(t+\tau)/2}$$

the coercivity inequality (2) was used with $p = \beta/(1-\beta)$, and to estimate the integral

$$\int_{(t+\tau)/2}^{t}$$

inequality (2) was used with $p = \alpha/(1-\alpha)$. Hence (4) follows at once. If now one uses the condition of Theorem 5 to estimate

$$\int_{\tau}^{(t+\tau)/2},$$

then, for $\alpha > 0$, one can obtain the inequality

$$|U(t,\tau)v|_\alpha \leqslant C|t-\tau|^{-\alpha}\|v\| + \varepsilon \left| U\left(\frac{t+\tau}{2}, \tau\right) v \right|_\alpha +$$

$$+\varepsilon|t-\tau|^{-1} \int_{\tau}^{(t+\tau)/2} \|U(s,\tau)v\|_\alpha \, ds.$$

From this (5) follows for $\alpha > 0$. To estimate $\|U(t,\tau)v\|$, one must use the estimates already obtained and the identity (6).

5. From (5) it follows that, under the conditions of Theorem 5, the operator-function $U(t,\tau)$ is strongly continuous jointly in the variables in E. Hence it follows (cf. (1), Theorem 3).

Theorem 6. The generalized solution in B_p of problem (3) for $F(t) \equiv 0$ has the form

$$v(t) = U(t,0)v_0 + Qf(t), \qquad \text{where} \quad Qf(t) = \int_0^t U(t,s)f(s)\, ds.$$

The smoothness properties of the operator $Qf(t)$ are described by the analogue of Theorem 4 from (1):

Theorem 7. Let $0 < \alpha,\ \beta < 1$, $0 \leq t \leq t + \Delta t \leq T$, $f(t) \in B_{1/\beta}$. Then

$$|Qf(t+\Delta t) - Qf(t)|_\alpha \leq \Delta t^{1-\beta-\alpha} C(\alpha,\beta) \left[\int_0^{t+\Delta t} \|f(s)\|^{1/\beta} \, ds \right]^\beta \qquad (\alpha+\beta < 1),$$

$$\left[\int_0^t |Qf(s)|_\alpha^{1/(\beta+\alpha-1)} \, ds \right]^{\beta+\alpha-1} \leq C(\alpha,\beta) \left[\int_0^t \|f(s)\|^{1/\beta} \, ds \right]^\beta \qquad (\alpha+\beta > 1).$$

If $\alpha + \beta \neq 1$, then on the left the norm $|w|_\alpha$ may be replaced by the norm $\|A^\alpha(0)w\|$.

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Note: Figure translations are in progress. See original paper for figures.