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Abstract

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MATHEMATICS

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ON THE COMPLETENESS OF THE SYSTEM OF EIGEN- AND ASSOCIATED ELEMENTS OF ONE CLASS OF NON-SELF-ADJOINT OPERATORS DEPENDING ON A PARAMETER λ

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The main purpose of the present article is to set forth the idea of a method that makes it possible to establish theorems on the completeness of the system of eigen- and associated (e.a.) elements of a broad class of operators that are rational and, in some cases, meromorphic functions of the parameter λ. These operators can be written in the form

$$A(\lambda) = \sum_{k=1}^{n} \sum_{i=1}^{n_k} (\lambda - a_k)^{-i} A_{i,k} + \sum_{j=0}^{r} \lambda^j A_j, \quad \text{where } n \leqslant \infty.$$

Along the way, some properties of the spectrum are also studied for a broader class of operators.

Let $A(\lambda)$ be an analytically λ-dependent completely continuous operator in a separable Hilbert space $\mathscr{H}$, and let a be a pole of the operator $A(\lambda)$, i.e. $A(\lambda)$ has the representation

$$A(\lambda) = \sum_{i=1}^{j} (\lambda - a)^{-i} A_i + B(\lambda),$$

where $B(\lambda)$ has no singularities in some neighborhood G_a of the point a.

Theorem 1. *If the operators A_i are finite-dimensional and $(E - A(\lambda))^{-1}$ exists at least at one point $\lambda_0 \in G_a$, then $(E - A(\lambda))^{-1}$ exists everywhere in G_a, with* *the exception of at most a countable set of points λ_i, which can only be poles of $(E - A(\lambda))^{-1}$ and have no limit points at any interior point of G_a.*

This theorem is very important for the further refinement of the well-known theorem of M. V. Keldysh ([1]), from which one can only conclude that the limit point of the eigenvalues of $A(\lambda)$ inside G_a can be only a.

Remark 1. Let us note that if at least one of the operators A_i is infinite-dimensional, then the theorem ceases to be true and the point a will be either a point of accumulation of poles or an essential singular point for $(E - A(\lambda))^{-1}$ (and it may be both in the sense that, for each element, one of these cases occurs).

Remark 2. The theorem remains true also in the case when $E - A(\lambda)$ decomposes into the sum of a finite-dimensional and an invertible operator.

Let $A(\lambda)$ be an operator analytically depending on λ and having as singular points only poles of finite orders (in the extended plane),

$$A(\lambda) = \sum_{i=1}^{n} \sum_{k=1}^{n_i} (\lambda - a_i)^{-k} A_{i,k} + \sum_{i=0}^{n_0} \lambda^i A_{0i}.$$

We shall assume that A_{i,n_i} $(i = 0, 1, \ldots, n)$ are complete, i.e. $\overline{A_{i,n_i}\mathcal{H}} = \mathcal{H}$. We form the system

$$Y_{i,h}^{j,r} = \sum_{k}^{h} \frac{1}{k!} \left[\frac{d^k}{d\lambda^k} \left(\psi_j^r(\lambda) \right) \right]_{\lambda = c_i} y_{i,\tilde{h}-k}, \quad r = 1, \ldots, n_j;\ j = 0, \ldots, n, \tag{1}$$

where $y_{i,h-k}$ is a chain of e.a. elements of the operator $A(\lambda)$, corresponding to the eigenvalue c_i, and $\psi_j^r(\lambda) = 1/(\lambda - a_j)^r$ for $j = 1, 2, \ldots, n$.

and $\psi_0^r(\lambda) = \lambda^r$. It is easy to see that if the operator A_i, r_i is complete, then the point a_i cannot be an eigenvalue of the operator $A(\lambda)$.

Following M. V. Keldysh, we shall call the system (1) a **derived chain** of eigen- and associated elements of the operator $A(\lambda)$. In the case

$$A(\lambda) = \sum_{i=0}^{j} \lambda^i A_i$$

the derived chains in the sense of this article differ from the derived chain in the sense of M. V. Keldysh (1) only by constant factors.

Definition. We shall say that a system of eigen- and associated elements of the operator $A(\lambda)$ is N-fold complete in $\mathcal{H}$, where

$$N = \sum_{j=0}^{n} n_j,$$

if the system

$$Y_{i,k} = \{Y_{i,k}^{j,r}\}_{\substack{r=1,\dots,n_i \\ j=0,\dots,n}}$$

is complete in the space $\mathfrak{B}_N$ ((2)), $k = 0, \dots, m_i$; $i = 1, 2, \dots$.

Let the operator $A(\lambda)$ have the form

$$A(\lambda) = \sum_{i=1}^{n} (\lambda - a_i)^{-1} A_i + \sum_{i=1}^{m} \sum_{k=1}^{n_i} (\lambda - b_i)^{-k} B_{i,k} T_i^k +$$

$$+ \sum_{i=1}^{l} \sum_{k=1}^{m_i} (\lambda - c_i)^{-k} C_{ik} + H(\lambda),$$

where $B_{ik} = E$ for $k = n_i$, and $B_{i,k}$ are arbitrary completely continuous operators for $k < n_i$; $H(\lambda)$ is one of the two operators $B + \lambda A_0$ or

$$B_{0,0} + \lambda B_{0,1} T_0 + \dots + \lambda^{n_0 - 1} B_{0,n_0 - 1} T_0^{n_0 - 1} + \lambda^{n_0} T_0^{n_0};$$

C_{ik} are arbitrary finite-dimensional operators.

Theorem 2. *Suppose that the following conditions are satisfied:*

1. The operators A_i $(i = 0, \dots, n)$, T_j $(j = 0, \dots, m)$ are complete operators of finite order ρ_i $(i = 0, \dots, n)$, r_j $(j = 0, \dots, m)$, respectively.

2. a) For each $0 \le i \le n$ there exists a system of rays G_i, issuing from the origin, such that the angle between neighboring rays of the system G_i is less than π/ρ_i, and on each ray of G_i, for sufficiently large $|\lambda|$,

$$\|(E - \lambda A_i)^{-1}\| \le c_i, \quad i = 0, \dots, n;$$

b) for each $0 \le j \le m$ there exists a system of rays R_j such that the angle between neighboring rays of the system R_j is less than π/r_j, and on each ray of R_j, for sufficiently large $|\lambda|$,

$$\|\lambda^t (E - \lambda^{n_j} T_j^{n_j})^{-1} T_j^t\| \le d_j \qquad (t = 0, \dots, n_j - 1;\ j = 0, \dots, m).$$

3.

$$\lim_{n\to\infty} \varepsilon_n(B) < \min_{i,j} \left\{ \frac{1}{c_i}, \frac{1}{d_j} \right\},$$

where $\varepsilon_n(B)$ is the best approximation of the operator B by n-dimensional operators.

4. $(E - L_a(\lambda))^{-1}$ exists and is bounded in some (arbitrary) neighborhood of the point a for $a = a_i$ $(i = 0, \dots, n)$, $a = b_i$ $(i = 0, \dots, m)$, where the operator $L_a(\lambda)$ is the principal part of $A(\lambda)$ in a neighborhood of the pole a.

Then the system of eigen- and associated elements of the operator $A(\lambda)$ is N-fold complete in $\mathcal{H}$, where

$$N = n + \sum_{i=0}^{m} n_i,$$

if

$$H(\lambda) = B_{0,0} + \lambda B_{0,1} T_0 + \dots + \lambda^{n_0} T^{n_0},$$

and

$$N = n + \sum_{i=1}^{m} n_i + 1,$$

if $H(\lambda) = B + \lambda A_0$.

The limiting points of the eigenvalues will be the points a_i $(i = 1, \dots, n)$, b_i $(i = 1, \dots, m)$, and $a_0 = \infty$, i.e. those poles of $A(\lambda)$ in whose neighborhoods the principal part is not degenerate.

For simplicity of exposition and clarity of the idea of the proof, we shall prove the theorem in the case when the poles are simple.*

* In (2), using this method, the author proved an analogous theorem for the case when there are two poles of derived orders (at the points 0 and ∞).

Consider the equation $y = A^*(\bar{\lambda})y + f$.
Using conditions 1 and 4 of Theorem 2, in the same way as this is done in ([2]), we prove that the resolvent of the operator $A(\lambda)$ in a neighborhood of the point a_i has order not exceeding ρ_i. Further, denoting $z_i = (\lambda - \bar{a}_i)^{-1}$, for any linear operator P we find:

$$y = (E - z_{iA} i^*)^{-1}(E - P) L_{a_i}(\bar{\lambda}) y + (E - z_{iA} i^*)^{-1} P L_{a_i}(\bar{\lambda}) y + (E - z_{iA} i^*)^{-1} f.$$

Assuming that P is the projection operator corresponding to the first j eigenvalues* of the operator A_i, $P = P_{i,j}$, we obtain:

$$(E - z_{iA}i^*)^{-1} P_{i,j} L_{a_i}(\bar{\lambda}) f = \sum_{r=1}^{j} \sum_{k=1}^{k_r} (\mu_{k,i} - z_i)^{-k} R^i_{j,k} L_{a_i}(\bar{\lambda}),$$

where $R^i_{j,k}$ are finite-dimensional operators, and $\mu_{k,i}$ are eigenvalues of A^*_i.

Further, in the same way as is done in [2,3], it is proved that if $\bar{z}_i$ lies on the rays of G_i and $|z_i|$ is sufficiently large, then

$$\|C_i(\lambda)\| \equiv \|(E - z_{iA}i^*)^{-1} L_{a_i}(\bar{\lambda})\| \le \varepsilon_i < 1.$$

Consequently, the resolvent of the operator $A(\lambda)$ coincides with

$$(E - C_i(\lambda))^{-1} (E - z_{iA}i^*)^{-1}$$

in some neighborhood of the point a_i, when z_i lies on the rays of G_i, and therefore is bounded for these values of z_i.

Now suppose that $Y^j_{i,k}$ is not $(n+1)$-fold complete in $\mathfrak{H}$; then $Y_{i,k}$ is not complete in $\mathfrak{B}_{n+1}$, and there exist $f_0, f_1, \dots, f_n$ such that

$$\sum_j (Y^j_{i,k}, f_j) = 0 \qquad (k = 1, \dots, k_i;\ i = 1, 2, \dots).$$

Taking

$$f(\lambda) = \sum_i (\lambda - \bar{a}_i)^{-1} f_i + \lambda f_0,$$

consider the equation

$$y = A^*(\bar{\lambda}) y + f(\lambda). \tag{2}$$

It is easy to find that

$$y(\lambda) = (E - A^*(\bar{\lambda}))^{-1} f(\lambda)$$

has no singularities other than isolated singular points at the points a_i and ∞. Taking into account the estimates of the resolvent in neighborhoods of these points and applying Theorem 1 from [2], we obtain that $y(\lambda)$ has only poles of finite orders at the points a_i and ∞. Writing

$$y(\lambda) = \sum_{i=1}^{n} \sum_{k=1}^{q_i} (\lambda - \bar{a}_i)^k \tilde{y}_{i,k} + \sum_{i=0}^{q_0} \lambda^i \tilde{y}_{0,i}$$

in equation (2) and comparing the coefficients of the principal parts, we find that

$$\tilde{y}_{i,k} = 0, \quad k = 0, \dots, q_i; \quad i = 0, \dots, n.$$

Consequently, $f(\lambda) \equiv 0$ and $f_i = 0$ $(i = 0, \dots, n)$; this proves the theorem.

Remark 1. Condition 2 of Theorem 2 is satisfied when the operators A_i are self-adjoint or normal with eigenvalues inside certain angles (see ([1–3])), or have a complete system of eigen- and associated elements forming a basis (see ([4])), or the values of the quadratic form (A_{if}, f) lie inside a sector of opening less than π/ρ_i (see ([6])), or when

$$A_i = \sum_{k=1}^{n_i} A_{i,k}, \qquad \|(E - A_i)^{-1}\| \leq \sum_k \|(E - A_{i,k})^{-1}\|,$$

and $(A_{i,k} f, f)$ lie in mutually nonintersecting sectors of opening less than π/ρ_i (for example, if $A_{i,k}$ are orthogonal for $k = 1, \dots, n_i$).

Remark 2. Theorem 2 is also valid for the operator $A(\lambda)$ when the number $l = \infty$ and

$$\|(E - L_0(\lambda))^{-1}\| \leq M_a$$

is satisfied only on circles contracting to the point a, for $a = a_i$ $(i = 0, \dots, n)$, $a = b_j$ $(j = 1, \dots, m)$.

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* Considering the equation $y = \lambda A_i^* y + f$, where f is orthogonal to all eigen- and associated elements of the operator A_i, we obtain that $y(\lambda)$ is an entire function of order $\rho' \leq \rho_i$, and, applying the Phragmén–Lindelöf theorem, we obtain that $y(\lambda)$ does not depend on λ; but from $y_0 = \lambda A y_0 + f$ it follows that $f = 0$; consequently, the A_i have complete systems of eigen- and associated elements.

Remark 3. Condition 4 of Theorem 2 and the condition stated in Remark 2 to Theorem 2 are, in essence, conditions relating the location of the points c_i and the norms of the operators $C_{i,k}$, but for lack of space we do not give more concrete conditions.

Remark 4. Let us note that in the case when

$$A(\lambda) = \lambda^{-1} A_0 + \sum_{i=0}^{n-1} \lambda^i B_i T^i + \lambda^n T^n$$

or

$$A(\lambda) = \lambda^{-1} A_0 + B_0 + \lambda A_1,$$

the condition $\overline{A_0 \mathfrak{H}} = \mathfrak{H}$ can be weakened by imposing an additional condition on B_0. Thus, for example, one may require that $(E - P) B_0 = 0$, where P is the projection operator of the subspace $\overline{A_0\mathfrak{H}}$. Then the derived system of e.p. elements will be complete in the closure of elements of the form

$$\{(E - B_0)A_0 f, g_1, \dots, g_n\};$$

in particular, if $B_0 = 0$, then in the subspace obtained by closure of elements of the form

$$\{Af, g_1, \dots, g_n\},$$

and for $n = 1$, $\{Af, g\}$. Consequently:

Theorem 2′. *Suppose the operators A_0 and A_1 have finite orders ρ_0 and ρ_1; A_1 is a complete operator, and there exists a system of rays G_0 and G_1 such that the angle between neighboring rays of the system G_i is less than π/ρ_i $(i = 0, 1)$, and on the rays of the system G_i, for sufficiently large $|\lambda|$, the condition*

$$\|(E - \lambda A_i)^{-1}\| < c_i$$

is satisfied. Then, if

$$\inf_{K \ni \mathfrak{K}} \|B_0 - K\| < \min(1/c_0,\, 1/c_1),$$

where $\mathfrak{K}$ is the set of all finite-dimensional operators satisfying the condition $Kf \in \overline{A_0\mathfrak{H}}$, then the derived system of e.p. elements of the operator

$$\lambda^{-1}A_0 + B_0 + \lambda A_1$$

is complete in the subspace of the space $\mathfrak{B}_2$ consisting of the closure of elements of the form $\{A_0 f, g\}$.

A more particular case of this theorem—the case $B_0 = 0$ under very strict restrictions on the operators A_0 and A_1—was proved by another method in ([8]).

In the papers ([8], [10]), boundary-value problems are indicated for abstract and differential operators containing the parameter λ in the boundary conditions, for which the completeness of the system of e.p. elements is easily reduced to the completeness of the system of e.p. elements of the operator

$$\lambda^{-1}A_0 + \lambda A_1,$$

but, as indicated in ([8]), because of the strict conditions imposed on A_0 and A_1 (A_0 and A_1 have an absolute trace), the theorem on the completeness of the system of e.p. elements of the operator

$$\lambda^{-1}A_0 + \lambda A_1$$

from ([8]) cannot be applied to these problems. Theorem 2′ of the present note, in the case $B_0 = 0$, applies to these problems and resolves the question of the completeness of the system of e.p. elements for the problems that reduce to the corresponding completeness problem for the operator

$$\lambda^{-1}A_0 + \lambda A_1,$$

when A_0 and A_1 are operators of finite order.

This method also makes it possible to prove the same theorems in a Banach space. For lack of space we do not state these theorems; we note only that Theorem 2 of the present article for operators of the form

$$A(\lambda) = \sum_{i}^{n} (\lambda - a_i)^{-1} A_i + B_0 + \lambda B$$

is also valid in the case when $\mathfrak{H}$ is a separable Banach space, if condition 3 in it is replaced by the condition

$$\lim_{n\to\infty} \varepsilon_n(B_0) = 0.$$

A finite-dimensional operator in a Banach space is defined as in Hilbert space ([5]). In proving analogous theorems in a separable Banach space, the theorems on estimates of the resolvent in Banach spaces from ([7]) are used.

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Note: Figure translations are in progress. See original paper for figures.