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Yu. I. NEIMARK, L. P. SHILNIKOV

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Abstract

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Yu. I. NEIMARK, L. P. SHILNIKOV

ON ONE CASE OF THE BIRTH OF PERIODIC MOTIONS

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Suppose that, for the dynamical system under consideration, depending on a parameter μ, when $\mu = 0$ there is a phase trajectory Γ_0 issuing from a simple saddle equilibrium state O and returning to it. We shall assume that the following conditions are satisfied:

1) In some domain G containing the equilibrium state O, the equations of motion are written in the form:

$$\dot{x}_i = \lambda_i(\mu) x_i \qquad (i = 1, 2, \dots, n), \tag{1}$$

where $\operatorname{Re}\lambda_m \leq \dots \leq \operatorname{Re}\lambda_2 \leq \operatorname{Re}\lambda_1 < 0 < \lambda_n < \operatorname{Re}\lambda_{n-1} \leq \dots \leq \operatorname{Re}\lambda_{m+1}$, and $\lambda_i(\mu)$ are sufficiently smooth functions of the parameter μ.

2) The phase trajectory Γ_0, as $t \to -\infty$, enters the saddle O, tangent to the leading axis Ox_n.

3) In the domain G the trajectory Γ_0 can be intersected by sufficiently smooth cross-section surfaces S $(S(x_1, \dots, x_m) = 0)$ and $\bar{S}$ $(x_n = d,\ d > 0)$ in such a way that the point mapping, generated by the phase trajectories of the system, of the surface $\bar{S}$ into S in neighborhoods of the points of intersection of $\bar{S}$ and S with Γ_0 exists and depends sufficiently smoothly on the variables $x_1, x_2, \dots, x_{n-1}$ and the parameter μ for all $|\mu| < \mu_0$, and $\partial S / \partial x_1 \neq 0$.

The problem of the present work is to clarify the conditions under which, when the parameter μ is varied from zero, a periodic motion is born from the phase trajectory Γ_0. The results obtained are formulated in Theorems 1–4*.

1. Denote by $M^0(x_1^0, \dots, x_m^0, 0, \dots, 0)$ and $\bar{M}^0(0, \dots, 0, \bar{x}_{m+1}^0, \dots, \bar{x}_n^0)$ the points of intersection of Γ_0 with the cross-section surfaces S and $\bar{S}$. Let $M(x_1, \dots, x_n)$ be a point on the surface S, sufficiently close to M^0. The point mapping of the surface S into $\bar{S}$, according to (1), can be written in the form

$$\bar{x}_j = x_j e^{\lambda_j \bar{t}} = x_j \left(\frac{x_n}{d}\right)^{-\lambda_j/\lambda_n} \qquad (j = 1, 2, \dots, n-1), \tag{2}$$

where $\bar{t}$ is the root of the equation $d = x_n e^{\lambda_n \bar{t}}$. We denote the mapping (2) by T_0. Obviously, T_0 is defined for sufficiently small $x_n > 0$. By assumption, the point $\bar{M}$ on the surface $\bar{S}$, lying in some ε-neighborhood of the point $\bar{M}^0$, is transformed into the point $\tilde{M}$ of the surface S so that

$$\tilde{x}_j = f_j(\bar{x}_1, \dots, \bar{x}_{n-1}, \mu) \qquad (j = 1, \dots, n), \tag{3}$$

where f_j are sufficiently smooth functions of their variables and $S(\tilde{x}_1, \dots, \tilde{x}_m) = 0$. We denote the mapping (3) of the surface $\bar{S}$ into S by T_1. Note that, for $\mu = 0$, $M^0 = T_1 \bar{M}^0$.

Consider the mapping $T = T_1 T_0$

$$\tilde{x}_j = f_j\left(x_1 \left(\frac{x_n}{d}\right)^{\nu_1}, \dots, x_{n-1} \left(\frac{x_n}{d}\right)^{-\nu_{n-1}}, \mu\right), \tag{4}$$

* For a second-order system, the conditions for the birth of a limit cycle from a separatrix loop were found in

where $v_j(\mu) = -\lambda_j \lambda_n^{-1}$ for $j = 1, 2, \dots, m$ and $v_j(\mu) = +\lambda_j \lambda_n^{-1}$ for $j = m + 1, \dots, n-1$ ($\operatorname{Re} v_j > 0$ for all $1 \le j \le n-1$) and $S(x_1, \dots, x_m) = 0$. The mapping T is defined for all points M for which $\rho(TM, \overline{M}^0) < \varepsilon$. Introduce new variables $\xi_1, \dots, \xi_n$ according to the formulas

$$x_i = x_i^0 + \xi_i, \qquad x_j \left(\frac{x_n}{d}\right)^{-v_j} = \bar{x}_j^0 + \xi_j, \qquad x_n = \xi_n, \tag{5}$$

$$i = 1, \dots, m; \quad j = m+1, \dots, n-1.$$

In the new variables $\xi_1, \dots, \xi_n$, the coordinates of the fixed points $M^*(\xi_1^*, \dots, \xi_n^*)$ of the transformation T will satisfy the system

$$\xi_i^* = f_i\left((x_1^0 + \xi_1^*)\left(\frac{\xi_n^*}{d}\right)^{v_1}, \dots, \bar{x}_{n-1}^0 + \xi_{n-1}^*; \mu\right) - x_i^0, \qquad i = 1, \dots, m;$$

$$(\bar{x}_j^0 + \xi_j^*)\left(\frac{\xi_n^*}{d}\right)^{v_j} = f_j\left((x_1^0 + \xi_1^*)\left(\frac{\xi_n^*}{d}\right)^{v_1}, \dots, \bar{x}_{n-1}^0 + \xi_{n-1}^*; \mu\right),$$

$$j = m+1, \dots, n-1; \tag{6}$$

$$\xi_n^* = f_n\left((x_1^0 + \xi_1^*)\left(\frac{\xi_n^*}{d}\right)^{v_1}, \dots, \bar{x}_{n-1}^0 + \xi_{n-1}^*; \mu\right).$$

The closeness of the point M^* to M^0 and of TM^* to $\overline{M}^0$ will hold if $\xi_n^* > 0$ and if $\xi_1^*, \dots, \xi_n^*$ are sufficiently small. Thus the problem of the birth of a periodic motion from the phase trajectory Γ_0 reduces to finding conditions under which the mapping T found by us has a fixed point M^* tending to M^0 as $\mu \to 0$.

2. Let us proceed to finding the fixed points of the transformation T. In view of the smallness of the quantities $\xi_1, \dots, \xi_n$, the system (6) can be written in the form

$$\sum_{p=1}^{m} \frac{\partial S(x_1^0, \dots, x_m^0)}{\partial x_p} \xi_p^* + \dots = 0,$$

$$\xi_i^* = A_i(\mu) + \sum_{p=1}^{m} B_{ip}(x_p^0 + \xi_p^*)\left(\frac{\xi_n^*}{d}\right)^{v_p} + \sum_{q=m+1}^{n-1} B_{iq}\xi_q^* + \dots, \qquad i = 2, \dots, m;$$

$$(\bar{x}_j^0 + \xi_j^*)\left(\frac{\xi_n^*}{d}\right)^{v_j} = A_j(\mu) + \sum_{p=1}^{m} B_{jp}(x_p^0 + \xi_p^*)\left(\frac{\xi_n^*}{d}\right)^{v_p} +$$

$$+ \sum_{q=m+1}^{n-1} B_{jq}\xi_q^* + \dots, \qquad j = m+1,\ n-1; \tag{7}$$

$$\xi_n^* = A_n(\mu) + \sum_{p=1}^{m} B_{np}(x_p^0 + \xi_p^*)\left(\frac{\xi_n^*}{d}\right)^{v_p} + \sum_{q=m+1}^{n-1} B_{nq}\xi_q^* + \dots,$$

where

$$A_k(\mu) = \begin{cases} f_k(0, \dots, 0, \bar{x}_{m+1}^0, \dots, \bar{x}_{n-1}^0; \mu) - x_k^0, & k = 2, \dots, m, \\ f_k(0, \dots, 0, \bar{x}_{m+1}^0, \dots, \bar{x}_{n-1}^0; \mu), & k = m+1, \dots, n; \end{cases}$$

$$B_{ks}(\mu) = \frac{\partial f_k(0, \dots, 0, \bar{x}_{m+1}^0, \dots, \bar{x}_{n-1}^0; \mu)}{\partial x_s}.$$

The Jacobian of the first $n-1$ equations of this system at the point $\xi_1^* = \dots = \xi_n^* = 0$ is nonzero for all sufficiently small μ, if

$$\Delta(\mu) = \begin{vmatrix} B_{m+1\,m+1} & \cdots & B_{m+1\,n-1} \\ \cdots & \cdots & \cdots \\ B_{n-1\,m+1} & \cdots & B_{n-1\,n-1} \end{vmatrix} \tag{8}$$

for $\mu = 0$ is different from zero. Assuming that $\Delta(0) \neq 0$, from the first $n-1$ equations (7) one can find $\xi_1^*, \dots, \xi_{n-1}^*$ in the form

$$\xi_j^* = \alpha_j(\mu) + \sum \varphi_{js} \left(\frac{\xi_n}{d}\right)^{\nu_s(\mu)} + \cdots, \qquad j = 1, \dots, n-1. \tag{9}$$

After substituting the found values $\xi_1^*, \dots, \xi_{n-1}^*$ into the last equation (7), we arrive at an equation for ξ_n^* of the form

$$\xi_n = \Phi_n(\mu) + \sum_{s=1}^{m} C_s(\mu) \left(\frac{\xi_n}{d}\right)^{\nu_s(\mu)} + O\left[\left(\sum_{s=1}^{m} \left(\xi_n^{2\nu_s} + \xi_n^2\right)\right)^{1/2}\right], \tag{10}$$

where

$$\Phi_n(\mu) = \Delta^{-1}(\mu) \begin{vmatrix} \Delta(\mu) & A_{m+1} \\ & \cdots \\ & A_{n-1} \\ B_{n\,m+1} \dots B_{n\,n-1} & A_n \end{vmatrix},$$

$$C_s(\mu) = \frac{x_s^0}{\Delta(\mu)} \begin{vmatrix} \Delta(\mu) & B_{m+1\,s} \\ & \cdots \\ & B_{n-1\,s} \\ B_{n\,m+1} \dots B_{n\,n-1} & B_{n\,s} \end{vmatrix}.$$

Let us note that in the case $m = n-1$, $\Phi_n(\mu) = A_n(\mu)$, where $A_n(\mu)$ is the n-th coordinate of the point of intersection with the surface S of the phase curve issuing from the equilibrium state O in the region $x_n > 0$.

To each solution $\xi_n^*(\mu)$ of equation (10) that tends to zero on the right as μ tends monotonically to zero there corresponds the birth of a periodic motion from Γ_0 under the reverse change of the parameter.

Case 1. Let $\operatorname{Re}\nu_s(0) > 1$ for $s = 1, \dots, m$. In this case equation (10) has the unique solution $\xi_n^* = \Phi_n(\mu) + \cdots$, tending to zero together with μ. This solution will correspond to a periodic motion if $\Phi_n(\mu) > 0$.

Case 2. Let $\operatorname{Re}\nu_1(0) < 1$. If $\nu_1(0)$ is real and, for sufficiently small $\mu > 0$ ($\mu < 0$), $\Phi_n(\mu)C_1(0) < 0$, then equation (10) has a unique solution tending to zero as $\mu \to 0$. On the contrary, when $\nu_1(0)$ is a complex number, the number of solutions of equation (10) increases without bound as $\mu \to 0$. However, in this case there are no solutions tending to zero as $\mu \to 0$, and therefore the birth of a periodic motion from the closed phase trajectory Γ_0 does not take place.

3. To investigate the stability of the fixed point M^* of the mapping T, we study the behavior of the roots of the characteristic equation

$$\chi(z) = \tag{11}$$

$$\begin{vmatrix} \bar{B}^*_{ip}\left(\frac{\xi^*_n}{d}\right)^{\nu_1} - \delta_{ip} z & B^*_{iq} & \sum_{s=1}^{m} B^*_{is}(x^0_s + \xi^*_s)\left(\frac{\xi^*_n}{d}\right)^{\nu_s - 1} \\ \bar{B}^*_{jp}\left(\frac{\xi^*_n}{d}\right)^{\nu_1} & B^*_{iq} - \delta_{jq}\left(\frac{\xi^*_n}{d}\right)^{\nu_p} z & \sum_{s=1}^{m} B^*_{js}(x^0_s + \xi^*_s)\left(\frac{\xi^*_n}{d}\right)^{\nu_s - 1} - (x^0_j + \xi^*_j)\left(\frac{\xi^*_n}{d}\right)^{\nu_j - 1} z \\ \bar{B}^*_{np}\left(\frac{\xi^*_n}{d}\right)^{\nu_1} & B^*_{nq} & \sum_{s=1}^{m} B^*_{ns}(x^0_s + \xi^*_s)\nu_s\left(\frac{\xi^*_n}{d}\right)^{\nu_s - 1} - z \end{vmatrix} = 0,$$

$$i = 2, \dots, m; \quad p = 2, \dots, m; \quad j = m+1, \dots, n-1; \quad q = m+1, \dots, n-1,$$

where

$$B^*_{ks} = \left(\frac{\partial f}{\partial x_s}\right)_{T_0 M^*}, \qquad \bar{B}^*_{jp} = \left(\frac{\xi^*_n}{d}\right)^{\nu_p - \nu_1} B^*_{jp} - B^*_{j1}\left[\frac{\partial S}{\partial x_p}\right]_{M^*}\left[\frac{\partial S}{\partial x_1}\right]^{-1}_{M^*}.$$

For $m = n-1$ and $\operatorname{Re}\nu_s(0) > 1$, all elements of the determinant $\chi(0)$ vanish together with μ, while the coefficient of z^{n-1} is equal to $(-1)^{n-1}$. In view of this, all roots of equation (11) tend to zero as $\mu \to 0$, and therefore in the case under consideration the periodic motion that is born is stable. Conversely, if $\nu_1(0) < 1$ and $C_1(0) \neq 0$, then the coefficient of z^{n-2} tends to infinity as $\mu \to 0$. Consequently, instability occurs in this case.

When $n - m - 1 > 0$, because the coefficient of z^{n-1} tends to zero as $\mu \to 0$, and because the coefficient of z^m is representable in the form $\Delta(0) + o(\mu)$ in the case when $\operatorname{Re}\nu_i > 1$, $i = 1, \dots, m$, and the coefficient of z^{m-1} is representable in the form

$$[\nu_1 C_1 + o(\mu)]\left(\frac{\xi^*_n}{d}\right)^{\nu_1 - 1},$$

when $\nu_1(0) < 1$, it follows that equation (11) has a root z for which $|z| \to \infty$ as $\mu \to 0$.

4. The results obtained can be formulated as the following theorems:

Theorem 1. If $-\operatorname{Re}\lambda_i(0) > \lambda_n(0)$ $(i = 1, 2, \dots, n-1)$, then for sufficiently small $\mu > 0$ $(\mu < 0)$, for which

$$f_n(0, \dots, 0, \bar{x}^0_{m+1}, \dots, \bar{x}^0_{n-1}, \mu) > 0,$$

there is born from Γ_0 only one periodic motion, and it is stable.

Theorem 2. If $-\operatorname{Re}\lambda_i(0) > \lambda_n(0)$ $(i = 1, \dots, m)$, $m < n-1$, and $\Delta(0) \neq 0$, then for sufficiently small $\mu > 0$ $(\mu < 0)$, for which $\Phi_n(\mu) > 0$, there is born from Γ_0 a unique and unstable periodic motion.

Theorem 3. If $-\lambda_1(0) < \lambda_n(0)$, $\Delta(0) \neq 0$, and $C_1(0) \neq 0$, then for sufficiently small $\mu > 0$ $(\mu < 0)$, for which $C_1(0)\Phi_n(\mu) < 0$, there is born from Γ_0 only one periodic motion, and it is unstable*.

Theorem 4. In the case when $\lambda_1(0)$ is complex, Γ_0 tends as $t \to \infty$ into the saddle O, touching the leading plane x_1x_2, and $C_1(0) \neq 0$, no birth of a periodic motion from Γ_0 takes place, despite the fact that in this case, for sufficiently small μ, there are periodic motions in a neighborhood of Γ_0, and their number grows without bound as $\mu \to 0$.

Research Physico-Technical Institute
at Gorky State University
named after N. I. Lobachevsky

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* Theorems 2 and 3 are also valid in the case when Γ_0 enters the saddle as $t \to -\infty$, touching not the leading coordinate axis corresponding to the simple root λ_j $(m+1 \leq j < n)$.

Note: Figure translations are in progress. See original paper for figures.