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**Abstract**

**Full Text**

**MATHEMATICS**

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# ON THE ASYMPTOTIC BEHAVIOR OF ORTHONORMAL POLYNOMIALS AT ONE-SIDED SINGULAR POINTS OF WEIGHT FUNCTIONS (ALGEBRAIC SINGULARITIES)

*(Presented by Academician S. N. Bernshtein on 4 VIII 1964)*

The hypothesis, put forward as early as 1921 by V. A. Steklov, that orthonormal polynomials (o.n.p.) grow without bound together with $n$ at interior points of the interval of orthogonality at which the weight function $\rho(x)$ vanishes ([1]), has not yet been fully confirmed. Nevertheless, a certain stock of information has already been accumulated that makes it possible to proceed to a more detailed discussion of it.

Thus, for the weight function $\rho(x) = |x|^\gamma,\ \gamma \geq 0,\ -1 \leq x \leq +1$, the author found exact values of the o.n.p. at the zero of the weight function ([2])

$$\hat{y}_n(0) = \begin{cases} (-1)^{n/2}\sqrt{\dfrac{2n+\gamma+1}{2}}\,\dfrac{\Gamma((n+\gamma+1)/2)}{(n/2)!\Gamma((\gamma+1)/2)}, & n = 0, 2, 4, \dots, \\ 0, & n = 1, 3, 5, \dots \end{cases} \tag{1}$$

The apparatus of difference-differential equations (d.d. equations) introduced by the author ([3]), together with the scheme of the method of model equations proposed by him ([4]), makes it possible to obtain asymptotic representations of o.n.p. on the entire interval of orthogonality, including singular points of weight functions.

As in the case (1), so also in all other cases of weight functions admitting d.d. equations for $y_n(x)$, it turns out that at points where the weight function has a zero of some algebraic order, i.e., $\rho(x_0) = 0$ and, as $x \to x_0$, $\rho(x) = O(|x - x_0|^\gamma)$, $-1 < x_0 < +1,\ \gamma \geq 0$, the corresponding o.n.p. grow without bound together with $n$, and the order of growth is equal to one half of the order of the zero of the weight function, i.e.

$$\hat{y}_n(x_0) = O(n^{\gamma/2}),$$

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provided only that $\hat{y}_n(x_0) \neq 0$.

Characteristics of the dependence of the order of possible growth of o.n.p. on the character of the singular points of weight functions are also given in the work of Ya. L. Geronimus ([5]). Unfortunately, from his comparatively general theorems one does not obtain the concrete order and the characteristic of the growth of $\hat{y}_n(x)$ at the very zero of the weight function.

The points at which the weight function vanishes may be one-sided; that is, it may happen that

$$\rho(x) = \begin{cases} O((x_0 - x)^{\gamma_1}), & x \to x_0 - 0,\ \gamma_1 \geq 0, \\ O((x - x_0)^{\gamma_2}), & x \to x_0 + 0,\ \gamma_2 \geq 0, \end{cases} \qquad \gamma_1 \neq \gamma_2 \tag{$2^1$}$$

or else $\gamma_1 = \gamma_2$, but together with the weight function at the point $x = x_0$ the polynomials $\hat{y}_n(x)$ also vanish for arbitrarily large indices $n$, i.e.

$$\rho(x_0) = 0, \qquad \hat{y}_n(x_0) = 0 \quad \text{for } n = n_1, n_2, \ldots \tag{$2^2$}$$

What, then, is the order of growth of the polynomials $\hat{y}_n(x)$, orthonormal with such a weight function, at the point $x_0 \in (-1, +1)$?

The apparatus of differential equations, it turns out, also makes it possible to take into account such characteristics of a zero of the weight function as ($2^1$) and ($2^2$), and leads us to a paradoxical phenomenon:

At zeros of the weight function $\rho(x)$ that are points of types ($2^1$) and ($2^2$), the corresponding orthonormal polynomials $\hat{y}_n(x)$ do not have a definite asymptotic behavior. The character of growth of these polynomials to the left and to the right of the zero of the weight function is different. Namely:

$$\hat{y}_n(x_0 - 0) = O(n^{\gamma_1/2}), \qquad \hat{y}_n(x_0 + 0) = O(n^{\gamma_2/2}), \qquad \gamma_1 \neq \gamma_2; \tag{$3^1$}$$

$$\hat{y}_n(x_0 - 0) = -\hat{y}_n(x_0 + 0) = O(n^{\gamma/2}), \qquad \hat{y}_n(x_0) = 0, \qquad n = n_1, n_2, \ldots. \tag{$3^2$}$$

A singularity of type ($2^2$) is possessed, for example, by the weight function

$$\rho(x) = |x^2 - x_0^2|^{\gamma}, \qquad -1 \leq x \leq +1, \quad 0 < x_0 < 1, \tag{$4^1$}$$

where

$$\gamma = \begin{cases} \gamma_1, & -x_0 < x < x_0, \\ \gamma_2, & -1 \leq x < -x_0, \quad x_0 < x \leq +1, \end{cases} \qquad \gamma_1 \neq \gamma_2$$

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and a singularity of type ($2^2$) is possessed by the function

$$\rho(x) = |x|^\gamma, \qquad -1 \le x \le +1, \quad \gamma \ge 0 \quad \text{for } \hat{y}_{2k+1}(x), \qquad k = 0, 1, 2, \dots . \quad (4^2)$$

First of all, let us establish that the polynomials

$$\hat{y}_n(x) = a_n(x^n + d_n x^{n-2} + \cdots),$$

orthonormal with the weight function $\rho(x) = |x^2 - x_0^2|^\gamma$ on the interval $[-1, +1]$, where $0 < x_0 < 1$ and $\gamma > -1$ does not change its value on this interval, satisfy the differential equation

$$(1 - x^2)(x^2 - x_0^2)^2 y'' + [(1 - x^2)\gamma - x^2 + x_0^2]2x(x^2 - x_0^2)y'$$

$$+[n(n + 1 + 2\gamma)(x^2 - x_0^2)^2 + Q_n(x^2 - x_0^2) - D_n]y = 0, \qquad (5)$$

where

$$Q_n = -n^2 + (1 + 2\gamma(x_0^2 - 1))n - 2d_n(2n - 1 + 2\gamma), \qquad (6)$$

$$D_n = \begin{cases} 0, & \hat{y}_n(x_0) \ne 0, \\ 4x_0^2\gamma(x_0^2 - 1), & \hat{y}_n(x_0) = 0, \end{cases}$$

and it is known that $d_n = -n/4 + d + O(1)$ ([6]).

The scheme for deriving equation (5) is as follows. By integration by parts over the limits from $-1$ to $+1$ (with preliminary multiplication by $|x^2 - x_0^2|^\gamma x^k$, $k = 0, 1, 2, \dots, n - 1$) we establish that

$$(1 - x^2)(x^2 - x_0^2)^2 \hat{y}_n''(x) + [(1 - x^2)\gamma - x^2 + x_0^2]2x(x^2 - x_0^2)\hat{y}_n'(x) \equiv$$

$$\equiv -P_4(x)\hat{y}_n(x). \qquad (7)$$

Then, representing $P_4(x) = A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0$, we verify that $A_3 = A_1 = 0$ (the left-hand side of (7) has the same parity as $\hat{y}_n(x)$), and $A_4 = n(n + 1 + 2\gamma)$.

Rewriting

$$P_4(x) = n(n + 1 + 2\gamma)(x^2 - x_0^2)^2 + Q_n(x^2 - x_0^2) + D_n$$

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from the identity (7), we find for $Q$ and $D_n$ the formulas (6).

Let now

$$\gamma = \begin{cases} \gamma_1, & -x_0 < x < x_0, \\ \gamma_2, & -1 \leqslant x < -x_0, \quad x_0 < x \leqslant +1, \end{cases}$$

i.e., the weight function has one-sided singular points of type ($2^1$). But, as before ($\rho(x)$ is even!), there remains $\tilde{y}_n(x) = x^n + d_n x^{n-2} + \cdots$, and by dividing (without remainder!) the polynomial

$$(1-x^2)(x^2-x_0^2)\tilde{y}_n''(x) + [(1-x^2)\gamma_i - x^2 + x_0^2]2x(x^2-x_0^2)\tilde{y}_n'(x), \qquad i = 1, 2,$$

by the polynomial $\tilde{y}_n(x)$, we find for the coefficients $P_4(x)$ the former values, given by formulas (5) and (6), but which now change depending on the change of $\gamma$ as $x$ passes through the values $\mp x_0$.

Consequently, the differential equation (5), obtained for the case of a weight function without one-sided singularities, when these are present in the weight function ($4^1$), will retain its form; only in place of $\gamma$, on the corresponding intervals of variation of $x$, either $\gamma_1$ or $\gamma_2$ will be taken.

This equation, as $x_0 \to 0$, passes into the differential equation of polynomials orthogonal with the weight function ($4^2$) for $\gamma = 2\gamma_2$, and, as $x_0 \to 1$, into the equation for ultraspherical polynomials orthogonal with the weight function $\rho(x) = (1-x^2)^{\gamma_1}$; for $\gamma_1 = \gamma_2 = 0$ this equation is satisfied by the Legendre polynomials.

From equation (5), according to our scheme of the method of reference equations ($^4$), we obtain the following asymptotic representations for the polynomials orthonormal with the weight function ($4^1$):

$$\hat{y}_n(x) = C_{in}(1-x^2)^{-1/4}|x-x_0|^{-\gamma_i/2}\sqrt{|\arcsin x - \arcsin x_0|}\times$$

$$\times J_\nu(\sqrt{n(n+1+2\gamma_i)}\,|\arcsin x - \arcsin x_0|)\left[1 + O\left(\frac{\ln n}{n}\right)\right], \qquad (8)$$

where

$$\nu = \begin{cases} \dfrac{\gamma_i - 1}{2}, & \text{if } \hat{y}_n(x_0) \neq 0, \\ \dfrac{\gamma_i + 1}{2}, & \text{if } \hat{y}_n(x_0) = 0, \end{cases} \qquad 0 \leqslant x \leqslant l < 1, \quad 0 < x_0 < l$$

$i = 1$ for $0 \leqslant x \leqslant x_0$, $i = 2$, for $x_0 \leqslant x \leqslant l$, $C_{in} = O(n^{1/2})$.

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Since for small $z$ it is known that

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)}[1+o(z)],$$

then, for any fixed $n$, in a sufficiently small neighborhood of the point $x = x_0$, from (8) we obtain the following asymptotic representations in the case when $\hat{y}_n(x_0) \neq 0$:

$$\hat{y}_n(x) = \overline{C}_{in}\,[n(n+1+2\gamma_i)]^{(\gamma-1)/2}\,[1+o(n(x-x_0))],$$

where $i=1,\ 0 \leqslant x < x_0;\ i=2,\ x_0 < x \leqslant l < 1,\ \overline{C}_{in} = O(n^{1/2})$.

Choosing sufficiently rapidly convergent sequences $x_{1,n} \uparrow x_0$ and $x_{2,n} \downarrow x_0$, we obtain from this, as $n \to \infty$,

$$\hat{y}_n(x_{1,n}) = \overline{C}_{1,n} n^{(\gamma_1-1)/2}[1+o(1)] = O(n^{\gamma_1/2}),$$

$$\hat{y}_n(x_{2,n}) = \overline{C}_{2,n} n^{(\gamma_2-1)/2}[1+o(1)] = O(n^{\gamma_2/2}).$$

This establishes the paradoxical phenomenon ($3^1$) in the asymptotics of the orthonormal polynomials under consideration.

For polynomials orthonormal with the weight function ($4^2$), we similarly find

$$\hat{y}_n(x) = C_n(1-x^2)^{-1/4}x^{-\gamma/2}\sqrt{\arcsin x}\,J_\nu\{\sqrt{n(n+\gamma+1)}\,\arcsin x\}\times$$

$$\times\left[1+O\left(\frac{\ln n}{n}\right)\right],$$

where

$$C_n = O(n^{1/2}), \quad 0 \leq x \leq l < +1;$$

$$\nu = \begin{cases} \dfrac{\gamma+1}{2}, & n = 1,3,5,\dots, \\ \dfrac{\gamma-1}{2}, & n = 0,2,4,\dots. \end{cases}$$

Hence, for any fixed $n = 1,3,5,\dots$, in a sufficiently small neighborhood of the point $x = 0$ we may write

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$$\hat{y}_n(x) = \bar{C}_n n^{(\gamma+1)/2} x[1 + o(nx)], \qquad \hat{y}_n(0) = 0. \tag{9}$$

Consequently, any sequence $\{x_n\}$, where $0 < x_n = O(1/n)$, leads us to the asymptotic formula

$$\hat{y}_n(x_n) = \bar{C}_n n^{(\gamma-1)/2}[1 + O(1)] = O(n^{\gamma/2}), \qquad \hat{y}_n(0) = 0, \quad n = 1, 3, \dots .$$

By virtue of the oddness of $\hat{y}_n x$ for $n = 1, 3, 5, \dots$, we obtain

$$\hat{y}_n(-x_n) = -\bar{C}_n n^{(\gamma-1)/2}[1 + O(1)] = O(n^{\gamma/2}).$$

Thus the paradoxical phenomenon ($3^2$) in the asymptotics of o.n.p. is established for the case of the weight function ($4^2$).

For weight functions of a more complicated form, nothing essentially new appears in the differential equations for o.n.p., and the singular points of weight functions of the form ($2^1$) and ($2^2$) become one-sided singular points of these equations. Thus, for example, for the weight function

$$\rho(x) = |x|^\gamma, \qquad \gamma = \begin{cases} \gamma_1, & -1 \le x < 0, \\ \gamma_2, & 0 < x \le +1, \end{cases}$$

the corresponding o.n.p. $\hat{y}_n(x) = a_n(x^n - S_n x^{n-1} + d_n x^{n-2} + \dots)$ satisfy the differential equation

$$(1 - x^2)x^2 y'' + [(1 - x^2)\gamma_i - 2x^2]xy' + [n(n + 1 + \gamma_i)x^2 + Q_{nx} +$$

$$+ D_n]y = -\frac{a_{n-1}}{a_n}(Q_{n+1} - Q_n)\hat{y}_{n-1}(x),$$

where

$$Q_n = S_n(2n + \gamma_i), \qquad D_n = n(n - 1) + 2(2n - 1 + \gamma_i)d_n - n\gamma_i,$$

and ([6])

$$a_n = 2^{nA}(1 + o(1)), \qquad S_n = \sigma + o(1), \qquad d_n = n/4 + d + o(1),$$

and the constants $A, \sigma, d$ do not depend on $n$.

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*Note: Figure translations are in progress. See original paper for figures.*

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