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Abstract

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FORMULATION AND METHODS OF SOLVING THE GENERAL BOUNDARY-VALUE PROBLEM OF OPERATOR THEORY FROM THE VIEWPOINT OF FUNCTIONAL ANALYSIS.

PROBLEMS OF CAUCHY AND DIRICHLET TYPE

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MATHEMATICS

1. Let R and R_1 be separable linear topological spaces (l.t.s.); let L be a linear operator mapping R onto R_1; let N_L be the set of all elements annihilated by L; assume N_L is closed and of dimension greater than zero.

The assertion "for the operator L a linear boundary-value problem is given" has a general logical aspect. Without entering here into a detailed discussion, in this section we indicate our conclusion concerning the appropriate form of the general formulation of this problem.

In such a problem there are given a certain class τ of boundary conditions of the same type and a subspace $R'_\tau \subseteq R_1$ of admissible right-hand sides of the equation $Lx = y$. A quasi-complement M to N_L will be called a maximal of L relative to $R' = LM$. It turns out that, in the general logical aspect, the class τ is completely characterized by a certain maximal M_τ relative to R'_τ. This assertion has the following circumstances in view.

For every $x \in N_L + M_\tau$ there is a unique decomposition $x = z + u$, $z \in N_L$, $u \in M_\tau$. The mapping $x \to z$ defines a linear operator P_{M_τ}, namely: $P_{M_\tau} x = z$. The operator P_{M_τ} proves to be closed in $N_L + M_\tau$. For every $z \in N_L$ the set $z + M_\tau$ is a closed linear manifold mapped by means of L one-to-one onto R'_τ.

In view of what has been said, the problem of finding a solution x of the system

$$Lx = y \quad \text{with the boundary condition} \quad P_{M_\tau} x = z, \tag{1}$$

where $z \in N_L$ and $y \in R'_\tau$ are given, has a unique solution.

For a fixed maximal M_τ, the totality τ of all boundary conditions is exhausted by running through all elements $z \in N_L$. If $z \in N_L$, but $y \notin R'_\tau$, then problem (1) has no solution. For any fixed $y \in R'_\tau$ and different z_1 and z_2 from N_L, problem (1) has different solutions.

Let L_{M_τ} denote the restriction of L to M_τ, and $L^{-1}_{M_\tau}$ the inverse operator (it maps $R'_\tau = LM_\tau$ one-to-one onto M_τ). We have

$$x = P_{M_\tau}x + L^{-1}_{M_\tau}Lx, \qquad x \in N_L + M_\tau. \tag{2}$$

It follows from (2) that the solution of (1) has the form $x = z + L^{-1}_{M_\tau}y$.

Problem (1) is the simplest aspect of the general linear boundary-value problem. It is clear that the requirement of continuity of the operator $L^{-1}_{M_\tau}$ is equivalent to the continuous dependence of the solution on the right-hand side y of the equation. We shall call a maximal M **principal** when the operator P_M is continuous; in this case the boundary datum z depends continuously on the solution x in problem (1).

In the elementary situation, when L is the operator of differentiation of functions of a real variable, M_τ is the set of those of them,

which vanish at the point τ of the real axis,

$$P_{M_\tau}x = x(\tau), \qquad L^{-1}_\tau = \int_\tau^t,$$

and equality (2) is the classical equality of analysis

$$x(t) = x(\tau) + \int_\tau^t x'(\xi)\,d\xi.$$

2. Let M_0 be a principal maximal subspace and N_L a complete null l.t.s. (every Cauchy sequence has a limit). Then the general form of a maximal subspace M relative to LM_0 is as follows:

$$M = \{x - Sx\}_{x \in M_0}, \tag{3}$$

where S is a closed linear operator from M_0 into N_L; moreover $R = N_L + M_0 = N_L + M$.

When N_L and M_0 are subspaces of type (F), all maximal subspaces relative to LM_0 are principal. When the maximal subspace M is given, one may choose $S = P_M$.

3. Illustration for the infinitesimal operator I of a strongly continuous semigroup $\{u(t)\}_{0 \le t < \infty}$ of linear bounded operators in a Banach space B. We assume $u(0) = 1$, D_I is dense in B, $\|u(t)\| \le C$, $0 \le t < \infty$ (C is a constant). Let B_∞ be the totality of those $x \in B$ for which there exists $u(+\infty)x = \lim_{t\to+\infty} u(t)x$; let B^0_∞ be the part of B_∞ where $u(+\infty)x = 0$. Then

$M_\infty = B^0_\infty \cap D_I$ is a principal maximal subspace for I on the space $B_\infty \cap D_I$, $D_\pi = IM_\infty$, $I^{-1}_{M_\infty} = -\Pi$, where Π is the potential operator of the semigroup.

To M_∞ there corresponds the boundary-value problem: $Ix = -y$ with $\lim_{t\to+\infty} u(t)x = x_0$, where $x_0 \in N_I$, $[N_I = u(+\infty)M_\infty]$, and $y \in D_\Pi$ are given. The solution, as is known, is

$$x = x_0 + (-I)^{-1}_{M_\infty} y = x_0 + \Pi y = x_0 + \int_0^\infty u(\tau)y\,d\tau.$$

4. Formal derivation of the annihilated space and of a maximal subspace of the difference of operators. Application. Let A and F be some linear operators with domains D_A and D_F; let M_0 be a maximal subspace for A; $FD_F \subseteq AM_0$, and let the operator $(1 - A^{-1}_{M_0}F)^{-1}$ exist on D_A with values in $D_A \cap D_F$. Formal transformations show that

$$N_{A-F} = (1 - A^{-1}_{M_0}F)^{-1}N_A$$

and the set

$$M_1 = (1 - A^{-1}_{M_0}F)^{-1}M_0$$

is, in the case of closedness, a maximal subspace for $A - F$. Moreover, the following holds:

$$z = (A - F)(1 - A^{-1}_{M_0}F)^{-1}A^{-1}_{M_0}z, \qquad z \in AM_0; \tag{4}$$

$$(A - F)^{-1}_{M_1}z = (1 - A^{-1}_{M_0}F)^{-1}A^{-1}_{M_0}z, \qquad z \in AM_0 \cap AM_1.$$

Retaining the notation of item 3, denote by $B_{(0,\infty)}$ the l.t.s. whose elements are all functions $\{y(t)\}_{0\le t<\infty}$ with values in B, continuous, continuously differentiable, and satisfying the condition

$$\lim_{\to\infty} \frac{\ln\|y(t)\|}{t} < \infty$$

(convergence is with respect to continuous variation of $\|y(t)\|$ and $\|y'(t)\|$ for each t, $0 \le t < \infty$).

Consider in $B_{(0,\infty)}$ the operators $\hat{A} = d/dt - I$ and F, generated by a bounded linear operator F in B. It can be shown that

$$\hat{A}^{-1}_{M_0}\tilde{z}(t) = \int_0^t u(t-\tau)z(\tau)\,d\tau$$

for $0 \leq t < \infty$ and $\{z(t)\} = \tilde{z} \in B_{(0,\infty)}$. Hence one obtains convergence of the series

$$\sum_{n=0}^{\infty} (A_{M_0}^{-1} F)^n \tilde{z}(t), \qquad 0 \leq t < \infty,$$

for $\tilde{z} \in B_{(0,\infty)}$; for $\tilde{z} \in D_A$ the sum is an element of D_A. In view of this, the rules mentioned are applicable for obtaining N_{A-F} and M_1, and formula (4). Hence it follows,

that the function $y(t) = (1 - A_{M_0}^{-1} F)^{-1} A_{M_0}^{-1} z(t)$, in the class of functions $B_{(0,\infty)}$, is the unique solution of the problem

$$(d/dt - I - F) y(t) = z(t), \qquad 0 \leq t < \infty, \qquad y(0) = 0;$$
$$\bar{z} = \{z(t)\} \in B_{(0,\infty)}.$$

It is known that $I_1 = I + F$ is infinitesimal for some semigroup of operators $\{u_1(t)\}_{0 \leq t < \infty}$. From (4) it easily follows that

$$u_1(t) z = (1 - A_{M_0}^{-1} F)^{-1} u(t) z, \qquad 0 \leq t < \infty, \qquad z \in B.$$

This is a conveniently surveyable form of the perturbation formula for the semigroup (2).

5. Construction of the main maximal, when R is a subspace of the space C_Q of functions that are material and continuous on a bicompactum Q; problems of Dirichlet type. We shall assume that L is a closed operator and N_L contains constants and separates the points of Q. Let Γ be the T-boundary of Q with respect to N_L.* We shall suppose that $\Gamma \neq Q$ and that in R there is a dense subset R' of those functions $y \in R$ for each of which there is an $x \in N_L$ such that $x(q) \equiv y(q)$ on Γ.

It turns out that the subset M_0 of all $y \in R'$ annulled on Γ is the main maximal.

The boundary problem corresponding to this maximal will be called a problem of Dirichlet type (it naturally generalizes the classical Dirichlet problem, in which Γ is the ordinary boundary of the domain).

The boundary conditions for this item can also be given in the form of linear equations on the T-boundary; boundary problems of this kind should naturally be regarded as boundary-value problems.

6. Taylor-type formula for a cascade of boundary problems. By a cascade of linear boundary problems we shall understand a set $\{L_k, N_{L_k}, M_k\}_{1 \leq k \leq n}$ for which the relation

$$N_{L_k} + M_k \subseteq L_{k+1}M_{k+1}, \qquad 1 \leq k \leq n-1.$$

is satisfied. Denote by R^I the collection of elements x for which

$$x \in N_{L_n} + M_n, \qquad L_k L_{k+1} \cdots L_{nx} \in N_{L_{k-1}} + M_{k-1}, \qquad 2 \leq k \leq n.$$

For $x \in R^I$ the following Taylor-type formula is valid:

$$x = P_{M_n}x + L^{-1}_{nM_n}P_{M_{n-1}}L_{nx} + \cdots + L^{-1}_{nM_n} \cdots L^{-1}_{2M_2}P_{M_1}L_2 \cdots L_{nx} + \\ + L^{-1}_{nM_n} \cdots L^{-1}_{1M_1}L_1 \cdots L_{nx}. \qquad (*)$$

When $L_n = L_0, \quad M_k = M_0, \quad 1 \leq k \leq n, \quad L_0^k x \in N_{L_0} + M_0 \subseteq L_0 M_0, \quad 0 \leq k \leq n-1$, formula (*) gives us

$$x = \sum_{k=0}^{n-1} (L^{-1}_{0M_0})^k P_{M_0} L_0^k x + (L^{-1}_{0M_0})^n L_0^n x. \qquad (**)$$

Formula (**) is a generalization of the usual Taylor formula (when L_0 is the differentiation operator).

7. To each $x \in R^I$ assign the set $\bar{x} = \{x, L_{nx}, \ldots, L_2 L_3 \cdots L_{nx}\}$. Let a function f be given that assigns to each $x \in R^I$ an element $f(\bar{x}) \in L_1 M_1$; the operators $\{L_k\}_{1 \leq k \leq n}$ are assumed closed. Suppose also that, for known z and y, for which L_{kz} and $L_k M_k^{-1} y$ are defined, we know how to find the latter.

It is required to find a solution of the system

$$\text{equations} \quad L_1 L_2 \cdots L_{nx} = f(\bar{x}) \qquad (K_I)$$

under the Cauchy boundary conditions

$$P_{M_n}x = z_0, \qquad P_{M_k}L_{k+1} \cdots L_{nx} = z_{n-k},$$

where $1 \leq k \leq n-1$, the elements $z_{n-k} \in N_{L_k}$, $1 \leq k \leq n$, are given.

* $q_0 \in \Gamma$ means: for any neighborhood $U(q_0)$ there is $x_u \in N_L$ such that

$$\max_{q \in Q/U(q_0)} x_u(q) < \max_{q \in Q} x_u(q) \ (^{3,4}).$$

By $(\mathrm{K}_{\mathrm{I}}^{y})$ we denote problem $(\mathrm{K}_{\mathrm{I}})$ in which the function f is replaced by the known element y from L_1M_1. The solution of $(\mathrm{K}_{\mathrm{I}}^{y})$ is obtained directly from formula $(*)$.

Denote

$$\zeta_0 = z_0 + L_{nM_n}^{-1} z_1 + \dots + L_{nM_n}^{-1} \cdots L_{2M_2}^{-1} z_{n-1}, \qquad F(y) =$$
$$= f(\zeta_0 + L_{nM_n}^{-1} \cdots L_{1M_1}^{-1} y).$$

It follows from formula $(*)$ that problem $(\mathrm{K}_{\mathrm{I}})$ is equivalent to the successive solution of the equation $y = F(y)$ and of problem $(\mathrm{K}_{\mathrm{I}}^{y})$.

Suppose that in problem $(\mathrm{K}_{\mathrm{I}})$ one has

$$N_{L_1} + M_1 \subseteq L_1M_1 \subseteq L_2M_2$$

and all spaces $\{L_kM_k\}_{2\le k\le n}$ lie in a certain Banach structure R_1*. Suppose, moreover, that the operators $\{L_{kM_k}^{-1}\}_{1\le k\le n}$ are structurally monotone and bounded, the operator $L_{1M_1}^{-1}$ is Volterra, and the function $f(x_0, x_1, \dots, x_{n-1})$ satisfies the following structural Lipschitz condition:

$$\big|f(x_0, x_1, \dots, x_{n-1}) - f(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-1})\big| \le C_f \sum_{j=0}^{n-1} |x_j - \tilde{x}_j|,$$

where C_f is a linear, bounded, structurally monotone operator commuting with $L_{1M_1}^{-1}$.

Under these conditions, problem $(\mathrm{K}_{\mathrm{I}})$ is reducible, by means of formula $(*)$, to an equation of the form $y = \varphi(y)$, solvable by the contraction principle by the method of successive approximations.

Problem $(\mathrm{K}_{\mathrm{I}})$ is a certain variant of a Cauchy-type problem.

8. Suppose that linear operators $\{L_k\}_{0\le k\le m}$ are given; M_0 is maximal for L_0; R^{II} is the space of elements x for which there exist sets

   $$\bar{x} = \{L_0^{n_0} L_1^{n_1} \cdots L_m^{n_m} x\}_{n_0<n,\ n_0+n_1+\dots+n_m\le n}$$

   of elements belonging to $N_{L_0} + M_0$; L_0 commutes with $\{L_k\}_{1\le k\le n}$ in R^{II}, and

   $$R^{\mathrm{II}} \subseteq L_0(M_0 \cap R^{\mathrm{II}}).$$

   Suppose that a function f is given, assigning to the set $\bar{x}$, for each $x \in R^{\mathrm{II}}$, an element $f(\bar{x})$ from $L_0(M_0 \cap R^{\mathrm{II}})$.

The second variant of the Cauchy-type problem is the problem of solving the system

$$L_0^n(x) = f(\bar{x}), \qquad (\mathrm{K}_{\mathrm{II}})$$
$$P_{M_0}x = z_0, \qquad P_{M_0}L_0^k x = z_k, \qquad 1 \le k \le n-1,$$

where $\{z_k\}_{0\le k\le n-1} \subseteq N_{L_0} \cap R^{\mathrm{II}}$ are given.

For problem ($\mathrm{K_{II}}$), reductions and a method of solution analogous to those mentioned above for problem ($\mathrm{K_I}$) are possible.

9. Let L, N_L, M, P_M have their former meaning; $N_L + M \subseteq LM$; $\{a_k\}_{0\le k\le n}$ are linear operators defined in LM, $a_n = 1$, and let

$$\Psi_k(L_M^{-1}) = \sum_{m=0}^{n} a_m (L_M^{-1})^{k-m}, \qquad 0 \le k \le n.$$

It turns out that if the operator $\Psi_n(L_M^{-1})$ has an inverse on LM, then one can propose, in a convenient form, the solution x of the problem

$$\sum_{k=0}^{n} a_k L^k x = y, \qquad P_M L^k x = z_k, \qquad 0 \le k \le n-1,$$

where $y \in LM$, $\{z_k\}_{0\le k\le n-1} \subset N_L$ are given. For the case $z_k = 0$, $0 \le k \le n-1$, one has

$$x = (L_M^{-1})^n [\Psi_n(L_M^{-1})]^{-1} y.$$

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CITED LITERATURE

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* $0 < x < y$ implies $\|x\| \le \|y\|$, and there is a constant a such that $\|\|x\|\| \le a\|x\|$.

Note: Figure translations are in progress. See original paper for figures.