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Abstract

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SOME PLANE PROBLEMS OF THERMOELASTICITY UNDER PERIODIC THERMAL ACTIONS

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In this note we consider certain plane problems of thermoelasticity under periodic thermal actions. The problems are considered as quasistatic; the coupling of the equations of the theory of elasticity and the theory of heat conduction is not taken into account. The necessary solutions of the problems of heat-conduction theory are obtained from (1, 3) or are considered in the note. The note gives the solution of certain plane problems of thermoelasticity and points out the analogy between these problems and the problems of the theory of bending of plates lying on an elastic Winkler foundation. In considering the problems of thermoelasticity, the concept of a thermoelastic displacement potential is used (4).

1. Consider a thin plate with non-heat-insulated faces, taking into account heat exchange with the surrounding medium; let χ be the circular frequency of temperature variation; h the thickness of the plate; β the coefficient of heat exchange; T_1 the temperature of the surrounding medium; λ the coefficient of thermal conductivity; c the heat capacity.

The differential equation of heat conduction

$$\lambda h \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} \right) = 2\beta(T - T_1) + hc\frac{\partial T}{\partial t}, \tag{1}$$

in which, without loss of generality, we set $T_1 = 0$, has in the case under consideration the solution

$$T = \varphi_1^*(r,\theta)\sin\chi t + \varphi_2^*(r,\theta)\cos\chi t = \varphi_1(\xi,\theta)\sin\chi t + \varphi_2(\xi,\theta)\cos\chi t, \tag{2}$$

where the functions $\varphi(\xi,\theta)$ satisfy the equation

$$\Delta\Delta\varphi - 2b_0\Delta\varphi + \varphi = 0; \tag{3}$$

$$\Delta = \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial}{\partial \theta^2}, \qquad \xi = \frac{r}{l_1}, \qquad l_1 = (\delta_2^4 + \delta^4)^{-1/4},$$

$$b_0 = \frac{\delta_2^2}{\sqrt{\delta_2^4 + \delta^4}}, \qquad \delta_2^2 = \frac{2\beta}{\lambda h}, \qquad \delta^2 = \frac{c\chi}{\lambda}.$$

If $b_0 < 1$, then the solution of the heat-conduction equation may be represented in the form

$$\begin{aligned} T(\xi, \theta; t) = \sum_{n=0}^{\infty} & [c_{1n}\tilde{u}_n(\xi) + c_{2n}\tilde{v}_n(\xi) + c_{3n}\tilde{f}_n(\xi) + c_{4n}\tilde{g}_n(\xi)] \\ & \times (A_n \cos n\theta + B_n \sin n\theta) \sin \chi t \\ & + \sum_{n=0}^{\infty} [-c_{1n}\tilde{v}_n(\xi) + c_{2n}\tilde{u}_n(\xi) - c_{3n}\tilde{g}_n(\xi) + c_{4n}\tilde{f}_n(\xi)] \\ & \times (A_n \cos n\theta + B_n \sin n\theta) \cos \chi t, \end{aligned} \tag{4}$$

where

$$\tilde{u}_n(\xi) + i\tilde{v}_n(\xi) = J_n(\xi e^{i\psi}), \qquad \tilde{f}_n(\xi) + i\tilde{g}_n(\xi) = H_n^{(1)}(\xi e^{i\psi}),$$

$$\psi = \operatorname{arc\,tg}\left(-\sqrt{1 - b_0^2}/b_0\right),$$

for $\psi = \pi/4$: $\tilde{u}_n(\xi) = u_n(\xi)$, $\tilde{v}_n(\xi) = v_n(\xi)$, $\tilde{f}_n(\xi) = f_n(\xi)$, $\tilde{g}_n(\xi) = g_n(\xi)$.

If a point heat source $Q \sin \chi t$ acts on an infinite plate, then, as shown in (3),

$$T = -\frac{Q}{4\lambda h}\,[\tilde{f}_0(\xi) \cos \chi t + \tilde{g}_0(\xi) \sin \chi t]. \tag{5}$$

Using the analogy between the heat-conduction problem under consideration and problems of the theory of bending of a plate on an elastic Winkler foundation, we write the solution of the problem for an infinite plate loaded by sources $q(\theta; t) = A_n \cos n\theta \sin \chi t$ along a circumference of radius $a = \alpha l_1$ in the following form:

for $\xi \leq \alpha$

$$T = -\frac{\pi\alpha A_n l_1}{2\lambda h}\{[\tilde{f}_n(\alpha)\tilde{u}_n(\xi) - \tilde{g}_n(\alpha)\tilde{v}_n(\xi)] \cos \chi t + [\tilde{f}_n(\alpha)\tilde{v}_n(\xi) + \tilde{g}_n(\alpha)\tilde{u}_n(\xi)] \sin \chi t\} \cos n\theta, \tag{6}$$

for $\xi \geq \alpha$

$$T = -\frac{\pi\alpha A_n l_1}{2\lambda h}\{[\tilde{u}_n(\alpha)\tilde{f}_n(\xi) - \tilde{v}_n(\alpha)\tilde{g}_n(\xi)]\cos\chi t + [\tilde{v}_n(\alpha)\tilde{f}_n(\xi) + \tilde{u}_n(\alpha)\tilde{g}_n(\xi)]\sin\chi t\}\cos n\theta. \tag{7}$$

II. 2. Let us find the thermoelastic displacement potential in the form

$$\Phi = \Phi_1 + \Phi_2\sin\chi t + \Phi_3\cos\chi t, \tag{8}$$

where Φ_1 is a particular solution of the differential equation

$$\Delta\Phi_1 = \frac{1+\sigma}{1-\sigma}\alpha^* l_1^2 T, \tag{9}$$

α^* is the coefficient of thermal expansion; σ is Poisson' s ratio; Φ_2, Φ_3 are harmonic functions. It is not difficult to show that

$$\Phi_1 = A_1^* T + A_2^* \Delta T, \tag{10}$$

where $A_1^* = 2b_0\alpha^* l_1^2(1+\sigma)/(1-\sigma)$, $A_2^* = -\alpha^* l_1^2(1+\sigma)/(1-\sigma)$.

Obviously, $\Phi_3 = 0$; the function Φ_2 is chosen quite simply; it has the same singularities as Φ_1, but of opposite sign. If the temperature field is described by expressions (6), (7), then

$$\Phi_1 = B\nabla_n T, \tag{11}$$

where ∇_n is the Bessel operator of index n,

$$B = \alpha^* l_1^2 \frac{1+\sigma}{1-\sigma}\frac{\pi\alpha}{2\lambda h}A_n;$$

$$\nabla_n\tilde{u}_n = -\tilde{u}_n\cos 2\psi_0 + \tilde{v}_n\sin 2\psi_0; \qquad \nabla_n\tilde{v}_n = -\tilde{u}_n\sin 2\psi_0 - \tilde{v}_n\cos 2\psi_0;$$

$$\nabla_n\tilde{f}_n = -\tilde{f}_n\cos 2\psi_0 + \tilde{v}_n\sin 2\psi_0, \qquad \nabla_n\tilde{g}_n = -\tilde{f}_n\sin 2\psi_0 - \tilde{g}_n\cos 2\psi_0, \qquad \psi_0 = \tfrac{1}{2}\arccos b_0.$$

The expression for the thermoelastic potential takes an especially simple form if the bases of the plate are thermally insulated; in this case:

for $\xi \leq \alpha$

$$\Phi_1 = B\{[-f_n(\alpha)u_n(\xi)+g_n(\alpha)v_n(\xi)]\sin\chi t+[f_n(\alpha)v_n(\xi)+g_n(\alpha)u_n(\xi)]\cos\chi t\}\cos n\theta; \tag{12}$$

for $\xi \geq \alpha$

$$\Phi_1 = B\{[-u_n(\alpha)f_n(\xi)+v_n(\alpha)g_n(\xi)]\sin\chi t+[v_n(\alpha)f_n(\xi)+u_n(\alpha)g_n(\xi)]\cos\chi t\}\cos n\theta; \tag{13}$$

$$\Phi_2 = 0;$$

for $\xi \leq \alpha$

$$\Phi_3 = \frac{2}{\pi}B\alpha^{-n}\xi^n\cos n\theta; \tag{14}$$

for $\xi \geq \alpha$

$$\Phi_3 = \frac{2}{\pi}B\alpha^{n}\xi^{-n}\cos n\theta. \tag{15}$$

From this it is not difficult to note the analogy between the problems of calculating plates on a Winkler elastic foundation and determining the thermoelastic displacement potential; this analogy also applies to the boundary conditions, which, for example, coincide in form in the case where a Winkler plate with rectilinear edges has hinged support, while in the problem of the theory of heat conduction the temperatures on rectilinear boundaries are prescribed, etc. If the heat sources are located in the phase $\sin\chi t$ or $\cos\chi t$, then, replacing them by the corresponding forces and calculating the corresponding plate on an elastic foundation, we obtain, up to a constant factor, the term Φ_1 in the same phase, while applying the Laplace operator to this expression gives, up to a constant factor, the term in the phase $\cos\chi t$ and $\sin\chi t$, respectively. This circumstance is essential, since the indicated analogy makes it possible in a number of cases to facilitate the determination of the thermoelastic displacement potential.

3. In conclusion, let us dwell on the plane problem for a simply connected finite domain; let us consider a plane disk with a thermally insulated base, or a cylinder whose temperature does not change along its generators. Suppose that the cross-section of the cylinder is bounded by a simple contour with a continuously rotating tangent. We shall assume that on the contour the temperature is prescribed

$$T(s) = f_1(s)\sin\chi t + f_2(s)\cos\chi t,$$

where s is the reduced arc coordinate of the contour. The solution of this problem is given in [1]; it has the form

$$T(\xi, \theta; t) = \int_C \mu_1(\sigma) K_1^*(\xi, \theta; \sigma; t)\, d\sigma + \int_C \mu_2(\sigma) K_2^*(\xi, \theta; \sigma; t)\, d\sigma, \tag{16}$$

$$K_1^*(\xi, \theta; \sigma; t) = [f_0'(\rho) \sin \chi t - g_0'(\rho) \cos \chi t] \cos(n_\sigma, \rho),$$

$$K_2^*(\xi, \theta; \sigma; t) = [g_0'(\rho) \sin \chi t + f_0'(\rho) \cos \chi t] \cos(n_\sigma, \rho),$$

where ρ is the distance between the point ξ, θ of the domain and the point of the contour with arc coordinate σ.

The functions $\mu_1(\sigma)$ and $\mu_2(\sigma)$ are determined from the integral equations

$$-2\mu_1(s) + \int_C \mu_1(\sigma) K_{1,1}^*(\sigma, s)\, d\sigma + \int_C \mu_2(\sigma) K_{2,1}^*(\sigma, s)\, d\sigma = f_2(s), \tag{17}$$

$$2\mu_2(s) + \int_C \mu_1(\sigma) K_{1,2}^*(\sigma, s)\, d\sigma - \int_C \mu_2(\sigma) K_{2,2}^*(\sigma, s)\, d\sigma = f_1(s), \tag{18}$$

$$K_{1,1}^*(\sigma, s) = -K_{2,2}^*(\sigma, s) = g_0'(\rho_{\sigma,s}) \cos(n_\sigma, \rho_{\sigma,s}),$$

$$K_{1,2}^*(\sigma, s) = K_{2,1}^*(\sigma, s) = f_0'(\rho_{\sigma,s}) \cos(n_\sigma, \rho_{\sigma,s}),$$

$\rho_{s,\sigma}$ is the distance between the contour points σ, s.

In this case the thermoelastic potential can be represented in the form

$$\Phi = -\frac{\alpha^*(1+\sigma) l_1^2}{1-\sigma} \int_C \mu_1(\sigma) \left[f_0'(\rho) \cos \chi t + \left\{ g_0'(\rho) - \frac{2}{\pi\rho} \right\} \sin \chi t \right] d\sigma +$$

$$+ \frac{\alpha^*(1+\sigma) l_1^2}{1-\sigma} \int_C \mu_2(\sigma) \left[f_0'(\rho) \sin \chi t - \left\{ g_0'(\rho) - \frac{2}{\pi\rho} \right\} \cos \chi t \right] d\sigma. \tag{19}$$

If the flux passing through the boundary of a simply connected domain is equal to

$$q(s) = f_1(s) \sin \chi t + f_2(s) \cos \chi t,$$

we set

$$T(\xi,\theta;t)=\int_C \mu_1(\sigma)K_3^*(\xi,\theta;\sigma;t)\,d\sigma+\int_C \mu_2(\sigma)K_4^*(\xi,\theta;\sigma;t)\,d\sigma, \tag{20}$$

where

$$K_3^*(\xi,\theta;\sigma;t)=f_0(\rho)\sin\chi t-g_0(\rho)\cos\chi t,$$

$$K_4^*(\xi,\theta;\sigma;t)=g_0(\rho)\sin\chi t+f_0(\rho)\cos\chi t,$$

$\mu_1(\sigma)$, $\mu_2(\sigma)$ are determined from Fredholm integral equations of the second kind, which, for brevity, we do not write out (see (2)). The thermoelastic potential has the form

$$\Phi_1=-\frac{\alpha(1+\sigma)l_1^2}{1-\sigma}\int_C \mu_1(\sigma)\left\{\left[g_0(\rho)-\frac{2}{\pi}\ln\rho\right]\sin\chi t+f_0(\rho)\cos\chi t\right\}d\sigma+$$

$$+\frac{\alpha(1+\sigma)l_1^2}{1-\sigma}\int_C \mu_2(\sigma)\left\{f_0(\rho)\sin\chi t-\left[g_0(\rho)-\frac{2}{\pi}\ln\rho\right]\cos\chi t\right\}d\sigma. \tag{21}$$

The generalization to the case in which the bases are not thermally insulated is obvious.

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References

[1] B. G. Korenev, *DAN*, **112**, No. 1 (1957).
[2] B. G. Korenev, *Problems in the Analysis of Beams and Plates on an Elastic Foundation*, Moscow, 1954.
[3] B. G. Korenev, *Some Problems of the Theory of Elasticity and Heat Conduction Solved in Bessel Functions*, Moscow, 1960.
[4] G. Parkus, *Nonstationary Thermal Stresses*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.