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ON FRACTIONAL POWERS OF ELLIPTIC OPERATORS

(Presented by Academician S. L. Sobolev on 26 IV 1965)

As has become clear in recent years, in a number of problems of functional analysis and mathematical physics (the study of linear and nonlinear elliptic and parabolic equations, Fourier series in eigenfunctions of elliptic operators, justification of the Fourier method, analysis of certain approximate methods, etc.) an important role is played by the theory of fractional powers A^τ of linear operators A (see, for example, ([1–8])). In this theory one of the basic questions is the question of from which spaces into which the operators A^τ act, if it is known from which spaces into which the operator A acts.

Let Ω be a bounded closed subset of a finite-dimensional Euclidean space. Below, $\lambda(x;h)$ denotes the measure of the set of those points $s \in \Omega$ at which $|x(s)| \geq h$.

In the article we use the Lebesgue spaces $\mathcal{L}_\alpha$, Lorentz spaces Λ_α, and Marcinkiewicz spaces M_α, $0 \leq \alpha \leq 1$ (see ([9,10])). For $\alpha = 0$ these spaces coincide with the space of essentially bounded functions:

$$\|x(s)\|_{\mathcal{L}_0} = \|x(s)\|_{\Lambda_0} = \|x(s)\|_{M_0} = \operatorname{vrai\,sup} |x(s)|.$$

The spaces $\mathcal{L}_\alpha$ and Λ_α for $0 < \alpha \leq 1$ are Banach spaces and consist of functions for which the corresponding norm is finite,

$$\|x(s)\|_{\mathcal{L}_\alpha} = \left\{ \int_\Omega |x(s)|^{1/\alpha}\, ds \right\}^\alpha, \qquad \|x(s)\|_{\Lambda_\alpha} = \int_0^\infty [\lambda(x;h)]^\alpha\, dh.$$

The space M_α for $0 < \alpha < 1$ is Banach and consists of functions with finite norm

$$\|x(s)\|_{M_\alpha} = \sup_{D \subset \Omega} (\operatorname{mes} D)^{\alpha - 1} \int_D |x(s)|\, ds.$$

The space M_1 is not Banach; it consists of functions with finite quasinorm

$$\|x(s)\|^*_{M_1} = \sup_{0<h<\infty} h\lambda(x;h).$$

The closure in M_α of the set of finite-valued functions will be denoted by M^0_α.

The sets of those points of the square $0 \le \alpha, \beta \le 1$ for which the linear operator A acts from $\mathcal{L}_\alpha$ into $\mathcal{L}_\beta$ and is continuous will be called (see ([11])) the $\mathcal{L}$-characteristic of the operator A and denoted by $\mathcal{L}(A; \text{cont.})$. The $\mathcal{L}$-characteristic $\mathcal{L}(A; \text{abs. cont.})$ is defined analogously. Obviously, each $\mathcal{L}$-characteristic, together with the point $\{\alpha_0, \beta_0\}$, contains all points $\{\alpha, \beta\}$ for which $0 \le \alpha \le \alpha_0$, $\beta_0 \le \beta \le 1$. Theorems on interpolation of continuity and complete continuity properties (see ([12,13])) mean that the $\mathcal{L}$-characteristics are convex.

The main aim of the present article is the construction of the $\mathcal{L}$-characteristics of fractional powers A^τ from the corresponding $\mathcal{L}$-characteristics of the operator A. It seems to us that the other results of the article are also of independent interest.

1. A closed linear operator A, acting in a Banach space E, is called positive if for all $\lambda < 0$ there exists the operator $(\lambda I - A)^{-1}$, defined on all of E, and if

$$\|\lambda(\lambda I - A)^{-1}\| < M \qquad (\lambda < 0).$$

In ([7,8]) positive and negative fractional powers A^τ of positive operators are defined and studied. We shall consider an operator A which acts in all the spaces $\mathcal{L}_\beta$ for $\beta \in (\beta_0, \beta_1)$ and is a positive operator. Denote by A^τ_β the fractional power of A as an operator in $\mathcal{L}_\beta$; it is easy to see that $A^\tau_{\beta''}$ is an extension of the operator $A^\tau_{\beta'}$, if $\beta'' > \beta'$. Therefore one may simply speak of the operator A^τ, omitting the index β.

We shall consider only fractional powers A^τ, where $0 < \tau < 1$. These fractional powers may be regarded as defined, for example, by the equality

$$A^\tau x = \frac{\sin \pi\tau}{\pi} \int_0^\infty \lambda^{\tau-1} (\lambda I + A)^{-1} A x \, d\lambda.$$

Theorem 1. *Let the operator A be positive in each space $\mathcal{L}_\beta$, $\beta \in (\beta_0, \beta_1)$. Let the curve*

$$\alpha = \eta(\beta) \qquad (\beta_0 < \beta < \beta_1), \tag{1}$$

where $\eta(\beta)$ is a nondecreasing function, lie in the $\mathcal{L}$-characteristic $\mathcal{L}(A;\ \text{discont.})$ or the $\mathcal{L}$-characteristic $\mathcal{L}(A;\ \text{abs. discont.})$.

Then the corresponding $\mathcal{L}$-characteristic $\mathcal{L}(A^\tau;\text{ discont.})$ or $\mathcal{L}(A^\tau;\text{ abs. discont.})$ of the operator A^τ, where $0<\tau<1$, contains all such points $\{\alpha,\beta\}$ that

$$\alpha < (1-\tau)\beta + \tau\eta(\beta). \tag{2}$$

Theorem 2. *Let the conditions of Theorem 1 be satisfied and let the curve (1) be a segment lying under the line $\beta=\alpha$.*

Then the corresponding $\mathcal{L}$-characteristic $\mathcal{L}(A^\tau;\text{ discont.})$ or $\mathcal{L}(A^\tau;\text{ abs. discont.})$ of the operator A^τ, where $0<\tau<1$, contains all points of the segment

$$\alpha = (1-\tau)\beta + \tau\eta(\beta) \qquad (\beta_0 < \beta < \beta_1). \tag{3}$$

The proofs of Theorems 1 and 2 use the moment inequalities for positive operators A (see ([8])):

$$\|A^\tau x\|_{\mathcal{L}_\beta} \le k(\beta)\|Ax\|^\tau_{\mathcal{L}_\beta}\|x\|^{1-\tau}_{\mathcal{L}_\beta}.$$

Next, the theorems formulated in the following item are applied, along with various interpolation theorems for properties of discontinuity and complete discontinuity (see ([9,12–16])).

2. Let E and E_1 be two arbitrary Banach spaces. By χ_D we shall denote the characteristic function of the set $D \subset \Omega$.

Theorem 3. *Suppose linear operators A, B on each characteristic function χ_D satisfy the inequality*

$$\|B\chi_D\|_E \le k\|A\chi_D\|^\tau_{E_1}\|\chi_D\|^{1-\tau}_{\mathcal{L}_\gamma}, \tag{4}$$

where $0<\tau<1$. Suppose A is continuous as an operator acting from the Lorentz space Λ_δ into E_1.

Then the operator B acts from the space $\Lambda_{\alpha(\tau,\gamma,\delta)}$, where

$$\alpha(\tau,\gamma,\delta) = \tau\delta + (1-\tau)\gamma, \tag{5}$$

into the space E and is continuous.

A subspace $F \subset E_1^*$ is called **determining** if, for some $a>0$,

$$\sup_{f\in F,\ \|f\|=1} |f(x)| \ge a\|x\| \qquad (x\in E_1).$$

Theorem 4. Suppose that the linear operators A, B, on every finite-valued function $x(s)$, satisfy the inequality

$$\|Bx\|_E \le k\|Ax\|_{E_1}^{\tau}\|x\|_{\mathfrak{L}_\gamma}^{1-\tau}, \qquad (6)$$

where $0 < \tau < 1$. Suppose that A is completely continuous as an operator acting from the space Λ_δ, where $0 < \delta < 1$, into E_1. Suppose that the values of the operator A^* adjoint to A, on some determining subspace $F \subset E_1^*$, lie in $M_{1-\delta}^0$.

Then the operator B acts from the space $\Lambda_{\alpha(\tau,\gamma,\delta)}$ into E and is completely continuous.

Theorem 5. Suppose that inequality (6) is fulfilled on functions of the form

$$x(s) = (\operatorname{mes} D_1)^{-\delta}\chi_{D_1}(s) - (\operatorname{mes} D_2)^{-\delta}\chi_{D_2}(s) \qquad (D_1, D_2 \subset \Omega). \qquad (7)$$

Suppose that A is completely continuous as an operator from Λ_δ into E_1, where $0 < \delta < \gamma \le 1$.

Then B is completely continuous as an operator from $\mathfrak{L}_\alpha$ into E for all $\alpha \in [0, \alpha(\tau,\gamma,\delta))$.

3. We shall call an operator B **symmetric** if $(Bx, y) = (x, By)$ for all finite-valued functions $x(s), y(s)$.

Theorem 6. Suppose that a symmetric operator B satisfies, on every characteristic function $\chi_D(s)$, the inequality

$$\|B\chi_D\|_{M_\beta} \le k\|A\chi_D\|_{E_1}^{\tau}\|\chi_D\|_{\mathfrak{L}_\gamma}^{1-\tau}, \qquad (8)$$

where A is a continuous operator acting from Λ_δ into E_1. Suppose that the inequalities $0 \le \gamma, \delta \le 1,\ 0 \le \beta < 1,\ \beta \le \alpha(\tau,\gamma,\delta)$ and $\alpha(\tau,\gamma,\delta) \ne 1-\beta$ are fulfilled.

Then B is continuous as an operator acting from each $\mathfrak{L}_{r(\lambda)}$ $(0 < \lambda < 1)$ into the corresponding $\mathfrak{L}_{q(\lambda)}$, where

$$r(\lambda) = (1-\lambda)\alpha(\tau,\gamma,\delta) + \lambda(1-\beta), \quad q(\lambda) = (1-\lambda)\beta + \lambda[1-\alpha(\tau,\gamma,\delta)]. \quad (9)$$

It is natural to expect that the complete continuity of the operator A implies the complete continuity of the operator B. We have been able to establish this fact only under certain additional assumptions.

4. We give one new theorem on interpolation of the property of complete continuity.

Theorem 7. Suppose that the operator A is completely continuous as an operator from Λ_{α_0} into M_{β_0} and continuous as an operator from Λ_{α_1} into M_{β_1}, where $0 \le \beta_0 \le \alpha_0 \le 1,\ 0 \le \beta_1 \le \alpha_1 \le 1,\ \alpha_0 \ne \alpha_1,\ \beta_0 \ne \beta_1$. Suppose that one of the conditions is fulfilled:

1) $0 \leq \beta < 1$ and $A\Lambda_{\alpha_0} \subset M^0_{\beta_0}$;

2) $0 < \alpha \leq 1$ and $A^*\Lambda_{1-\alpha_0} \subset M^0_{1-\beta_0}$.

Then A is completely continuous as an operator from $\mathfrak{L}_{\alpha(\lambda)}$ into $\mathfrak{L}_{\beta(\lambda)}$, where $0 < \lambda < 1$ and

$$\alpha(\lambda) = (1-\lambda)\alpha_0 + \lambda\alpha_1, \quad \beta(\lambda) = (1-\lambda)\beta_0 + \lambda\beta_1.$$

5. Let Ω be a bounded domain in n-dimensional space with sufficiently smooth boundary Γ. Let C be an elliptic operator defined by the differential expression

$$\mathcal{C}u(x) = \sum_{0\leq|r|\leq 2k} a_r(x)D^r u(x) \tag{10}$$

and boundary conditions of general form

$$\mathfrak{B}_j u(x) = \sum_{0\leq|r|\leq m_j} b_{rj}(x)D^r u(x) \qquad (j = 1, \dots, k;\ m_j \leq 2k-1).$$

Here, as usual, $r = (r_1, \dots, r_n)$; $|r| = r_1 + \dots + r_n$; D^r is a derivative generalized in the sense of S. L. Sobolev ([17]). In what follows it is assumed that the operator C is positive in each space $\mathcal{L}_a$, $0 < a < 1$. Conditions for the positivity of elliptic operators are indicated in ([18,19]). By T^0_ν $(0 < \nu < 1)$ we shall denote the part of the unit square $0 \leq \alpha, \beta \leq 1$ consisting of the points lying above the line $\beta = \alpha - \nu$ $(0 \leq \alpha < 1)$; and by T_ν, the union of T^0_ν with the segment $\beta = \alpha - \nu$ $(0 \leq \alpha < 1)$.

Let $2k < n$. From the embedding theorems of S. L. Sobolev ([17]) and the coercivity inequalities ([19]) it follows that the $\mathcal{L}$-characteristic $\mathcal{L}(C^{-1}$; discontinuous) contains the set $T_{2k/n}$, while the $\mathcal{L}$-characteristic $\mathcal{L}(C^{-1}$; completely discontinuous) contains the set $T^0_{2k/n}$. From the general Theorems 1 and 2 there follows the validity of an analogous assertion for the operators $C^{-\tau}$.

Theorem 8. *Let $2k < n$. Then the $\mathcal{L}$-characteristics $\mathcal{L}(C^{-\tau}$; discontinuous) and $\mathcal{L}(C^{-\tau}$; completely discontinuous) of the fractional powers $C^{-\tau}$ $(0 < \tau < 1)$ of the positive elliptic operator C contain, respectively, the sets $T_{2k\tau/n}$ and $T^0_{2k\tau/n}$.*

Stronger assertions for self-adjoint elliptic operators of second order $(k = 1)$ are contained in the papers of P. E. Sobolevskii ([20,21]), where estimates of Green' s functions and their derivatives are obtained for fractional powers $C^{-\tau}$ of elliptic operators C of second order. These estimates make it possible to study the operators $C^{-\tau}$ also in families of function spaces different from $\mathcal{L}_a$.

The assertion of Theorem 8 concerning the $\mathcal{L}$-characteristic $\mathcal{L}(C^{-\tau}$; completely discontinuous) was obtained by V. P. Glushko by another method.

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Note: Figure translations are in progress. See original paper for figures.