ON THE CONSTRUCTIVE THEORY OF THE RIEMANN INTEGRAL

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Abstract

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MATHEMATICS

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This note is a continuation of the work ([7]). We shall adhere to the conventions adopted in the introductory part of ([7]). In particular, by a "function" we shall everywhere (with the exception of Remark 2) mean a constructive function defined everywhere on the segment $0\Delta 1$. By δ and π we shall, as before, denote certain fixed algorithms that transform every integral sum respectively into its value and its mesh.

Let us adopt the following further definitions. A sequence S of integral sums of a function f is called **regularly contracting** if, for every n, one has $\pi(S(n)) < 2^{-n}$. A word of the form $r_0\square r_1\square \dots \square r_{n-1}\square r_n$, where r_m $(m = 0, \dots, n)$ are rational numbers such that $r_0 = 0$, $r_n = 1$, and $r_0 < r_1 < \dots < r_{n-1} < r_n$, we call a **rational partition**.

Let

$$S_0 \doteq \xi\{\square P\square u$$

be an integral sum of the function f, and

$$P \doteq r_0 * x_0 * r_1 * x_1 * \dots * r_{n-1} * x_{n-1} * r_n.$$

Then the rational partition $r_0\square r_1\square \dots \square r_{n-1}\square r_n$ we call the **rational partition corresponding to the integral sum** S_0. One can construct an algorithm D that transforms every integral sum into the rational partition corresponding to it.

1. The definition of Riemann integrability (R-integrability) given in ([7]) is the natural analogue of the classical ε-δ definition. Below we shall refer to this definition as Definition 1 of R-integrability. At the same time, in classical analysis a definition of R-integrability in the "language of sequences" is used. The analogue of such a definition is the following definition.

Definition 2 of R-integrability. A function f is called **R-integrable** on $0\Delta 1$ if one can construct an algorithm σ such that, for every regularly contracting sequence S of integral sums of the function f, the algorithm σ is applicable to $\xi S\}$ and transforms $\xi S\}$ into a record of the convergence regulator of the sequence of FR-numbers $(\delta \circ S)$.*

Functions R-integrable on $0\Delta 1$ by virtue of Definition 1 will be called R-integrable in sense 1. Functions R-integrable on $0\Delta 1$ by virtue of Definition 2 will be called R-integrable in sense 2.

Theorem 1. *Every function R-integrable on $0\Delta 1$ in sense 2 is R-integrable on $0\Delta 1$ in sense 1.*

Since, on the other hand, it is obvious that every function R-integrable on $0\Delta 1$ in sense 1 is R-integrable on $0\Delta 1$ in sense 2, it follows, by Theorem 1, that Definitions 1 and 2 of R-integrability are equivalent.

Remark 1. The known classical proof of the assertion corresponding to Theorem 1 is a pure existence proof (cf. ([2]), p. 118 and ([3]), p. 96), whereas Theorem 1 should be understood in the following way: one can construct an algorithm transforming

* $(\delta \circ S)$ denotes the composition of the algorithms S and δ (see ([1])).

any word of the form $\xi f\}\square\xi\sigma\}$, where f and σ satisfy Definition 2, into a record of a regulator of integrability of the function f.

The proof of Theorem 1 has some similarity with the proof of the main theorem in ([4]) and rests on the following lemmas.

Lemma 1. *In order that a function f be R-integrable on $0\Delta 1$ in the sense of 1, it is necessary and sufficient that there exist a sequence of rational partitions T and an algorithm λ of type $(\mathbb{N} \to \mathbb{N})$ such that, for every k and any integral sums S_1 and S_2 of the function f satisfying the condition $D(S_1) \doteq D(S_2) \doteq T(\lambda(k))$, the inequality $|\delta(S_1) - \delta(S_2)| < 2^{-k}$ holds.*

Lemma 2. *One can construct an algorithm $\mathfrak{C}$ such that: 1) for every word $m\square\xi f\square T_1\square u$, where m is a natural number, T_1 is a rational partition, f is a function, u is an FR-number, from $!\mathfrak{C}(m\square\xi f\}\square T_1\square u)$ it follows that $\mathfrak{C}(m\square\xi f\}\square T_1\square u)$ is an integral sum of the function f, $D(\mathfrak{C}(m\square\xi f\}\square T_1\square u)) \doteq T_1$ and $|\delta(\mathfrak{C}(m\square\xi f\}\square T_1\square u)) - u| > 2^{-m}$; 2) if there exists an integral sum S_1 of the function f satisfying the conditions $D(S_1) \doteq T_1$ and $|\delta(S_1) - u| > 2^{-m}$, then $!\mathfrak{C}(m\square\xi f\}\square T_1\square u)$.*

In the proof of Theorem 1 the lemma $\{6\}$([4]) is also used essentially.

Remark 2. Following the pattern given above, it is easy to give definitions, analogous to 1 and 2, for the case of an arbitrary segment of integration $x\Delta y$.*

Theorem 1 remains valid for constructive functions R-integrable on an arbitrary segment $x\Delta y$, i.e. one can construct an algorithm that transforms every word of the form $x\Delta y\square\xi f\}\square\xi\sigma\}$, where f and σ satisfy the correspondingly modified Definition 2, into a record of a regulator of integrability of f on $x\Delta y$.

2. We shall say that an F-number q is **conditionally equal** to an FR-number z if every FR-number with the same basis as q is equal to z. When these conditions are fulfilled we shall also say that z is conditionally equal to q.

It is easy to see that for every R-integrable function one can construct an F-number conditionally equal to its R-integral.

There is, in a certain sense, a converse assertion.

Theorem 2. *One can construct an algorithm transforming every F-number into a record of a function R-integrable on $0\Delta 1$ with R-integral conditionally equal to this F-number.*

Denote by Φ an exact disjoint 1/2-bounded segment covering of $0\Delta 1$ (see ([5]), Theorem 2.3).

Lemma 3. *One can construct an algorithm $\mathfrak{G}$ such that for every n: 1) $\mathfrak{G}_{n\square}$ is a function R-integrable on $0\Delta 1$; 2) $\mathfrak{G}_{n\square}$ vanishes on the segments $\Phi_0, \dots, \Phi_n$; 3) for every x from $0\Delta 1$ one has $0 \le \mathfrak{G}_{n+1\square}(x) \le \mathfrak{G}_{n\square}(x) \le 1$; 4) the R-integral of the function $\mathfrak{G}_{n\square}$ is not less than $1/4$.*

Let now $\mathfrak{A}$ be an arbitrary sequence of rational numbers. Denote by c_n the R-integral of the function $\mathfrak{G}_{n\square}$ and consider the series

$$\mathfrak{A}(0)\frac{1}{c_0}\mathfrak{G}_{0\square}(x) + \sum_{k=0}^{\infty}(\mathfrak{A}(k+1) - \mathfrak{A}(k))\frac{1}{c_k}\mathfrak{G}_{k\square}(x).$$

Using properties 2) and 4) of the algorithm $\mathfrak{G}$, it is easy to show that this series converges for every x from $0\Delta 1$ (cf. ([5]), Theorem 3.1). Suppose that the sequence $\mathfrak{A}$ is fundamental. Then, using property 3) of the algorithm $\mathfrak{G}$ and the inequalities $1/4 \le c_n$ and $c_{n+1} \le c_n$, which follow from properties 3) and 4), with the aid of estimates connected with the Abel transformation (see ([3]), p. 308, lemma), one can show that the series under consideration converges uniformly on $0\Delta 1$.

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* It is not excluded that $x = y$.

From these remarks, and also from the fact that in constructive analysis the theorem on termwise integration of a uniformly convergent series of R-integrable functions is valid, Theorem 2 follows easily.

Theorem 2 is a strengthening of Theorem 2 from (7).

3. Let us recall that a function f is called **uniformly continuous** on $0\Delta 1$ if one can construct an algorithm ρ of type $(\mathbb{N} \to \mathbb{N})$ such that

$$\forall n \forall xy(((x, y \in 0\Delta 1) \;\&\; (|x - y| < 2^{-\rho(n)})) \supset (|f(x) - f(y)| < 2^{-n})).$$

Let $z_1 \Delta z_2$ be a segment contained in $0\Delta 1$ (i.e. $0 \le z_1 \le z_2 \le 1$)*. We shall say that the function f is **effectively nonuniformly continuous** on $z_1 \Delta z_2$ if there exist a natural number k and sequences of FR-numbers α_1 and α_2 such that

$$\forall n\ ((\alpha_1(n),\ \alpha_2(n) \in z_1 \Delta z_2)\ \&\ (|\alpha_1(n) - \alpha_2(n)| < 2^{-n})\ \&$$
$$\&\ (|f(\alpha_1(n)) - f(\alpha_2(n))| \ge 2^{-k})).$$

We shall say that the function f is **everywhere effectively nonuniformly continuous** on $0\Delta 1$ if it is effectively nonuniformly continuous on every non-degenerate segment contained in $0\Delta 1$.

From Lemma 1 it follows in an obvious way that every function uniformly continuous on $0\Delta 1$ is R-integrable on $0\Delta 1$. As was noted in (7), there also exist functions effectively nonuniformly continuous on $0\Delta 1$ that are R-integrable on $0\Delta 1$.

I. D. Zaslavskii and G. S. Tseitin constructed an example of a function everywhere effectively nonuniformly continuous on $0\Delta 1$ ((5), Theorem 3.5). It is not difficult to show that the Zaslavskii–Tseitin function is effectively not Riemann integrable on $0\Delta 1$. In this connection there arises the question of the existence of such R-integrable functions. A positive answer to this question is given by the following theorem.

Theorem 3. *One can construct a function everywhere effectively nonuniformly continuous on* $0\Delta 1$ *and* R*-integrable on* $0\Delta 1$.

The construction of a function satisfying Theorem 3 is based on certain properties of the construction of I. D. Zaslavskii ((6), Theorems 4.2 and 5.2). In contrast to Theorem 3.5 from (5), the apparatus of singular coverings is not used.

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* $z_1 \leq z_2$ is understood as an abbreviation for the assertion "it is false that $z_1 > z_2$."

Note: Figure translations are in progress. See original paper for figures.

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