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G. L. LUKANKIN

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Abstract

Full Text

G. L. LUKANKIN

ON THE BEHAVIOR OF AN INTEGRAL OF TEMLYAKOV TYPE OF THE FIRST KIND AT THE SKELETON POINTS OF A DOMAIN D OF TYPE A

(Presented by Academician M. A. Lavrent' ev, 17 VIII 1964)

In the present paper, as domains D we consider domains of type A, i.e. domains of the form
$D = \{w, z : c|w| + d|z| < 1,\ c > 0 \text{ and } d > 0\}$, whose boundary
$\partial D = \{w, z : c|w| + d|z| = 1\}$ is represented in the absolute quadrant-plane by a straight line.

In what follows we shall consider the domains

$$D_+^+ = \{w, z : c|w| + d|z| < 1\}, \qquad D^{\mp} = \{w, z : c|w| - d|z| > 1\},$$

$$D^{\pm} = \{w, z : d|z| - c|w| > 1\},$$

$$D^{=} = \{w, z : c|w| + d|z| > 1, \quad |c|w| - d|z|| < 1\}$$

(their representation in the absolute quadrant-plane is shown in Fig. 1). Functions defined in these domains will be denoted respectively by

$$f_+^+(w, z), \qquad f^{\mp}(w, z), \qquad f^{\pm}(w, z) \quad \text{and} \quad f^{=}(w, z).$$

As is known ([4–6]), the Temlyakov integral of the first kind

$$f(w, z) = (4\pi^2 i)^{-1} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} F(\tau, t, \zeta)(\zeta - w)^{-1}\, d\zeta \tag{1}$$

($u = cw + dze^{it}$, and $F(\tau, t, \zeta)$ is an arbitrary Lebesgue-summable function prescribed on the boundary ∂D) is an analytic function in the domains D_+^+, $D^{\mp}$, $D^{\pm}$ and is not analytic in the domain $D^{=}$; however, under certain restrictions ([5,6]) on the density $F(\tau, t, \zeta)$, the integral (1) is continuous in the whole space C^2 of two complex variables w and z, with the exception of the sets

$$\mathfrak{A} = \{w, z : \ |w| = c^{-1},\ |z| = 0,\ c > 0\},$$

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$$\mathfrak{B} = \{w, z : \ |w| = 0, \ |z| = d^{-1}, \ d > 0\},$$

which have been called the skeleton of the domain D of type A. Subsequently it was found ([5,6]) that at the skeleton points of a domain D of type A, for the function $f(w, z)$ represented in the form of the integral (1) and having angular limiting values as the point (w, z) tends to a skeleton point from inside the domain D_+^+ and from inside the domains $D^\mp$ and $D^\pm$ (i.e. along paths located in the domains of analyticity of the integral (1)), formulas analogous to the Sokhotski formulas are valid.

Fig. 1

In the present paper we study the question of the behavior of the function $f(w, z)$ ($f(w, z)$ is a function represented by an integral of the indicated type) as the point (w, z) approaches the skeleton points from inside the domain $D^=$ (i.e. along paths located in the domain of nonanalyticity of the integral (1)). It turned out that the function $f(w, z)$ has no limit when approaching a skeleton point from inside the domain $D^=$, but has limiting values along paths located in any nonanalytic hyperplane passing through skeleton points and belonging to the domain $D^=$.

We shall say that $F(\tau, t, \xi) \in \delta$ if $F(\tau, t, \xi)$, as a function of ξ, satisfies a Lipschitz condition $\operatorname{Lip}\alpha$ $(0 < \alpha \leqslant 1)$, independent of τ and t (i.e.,

$$|F(\tau, t, \xi) - F(\tau, t, \xi_0)| < K|\xi - \xi_0|^\alpha,$$

where K and α do not depend on τ and t), and is representable on the boundary ∂D of the domain D by the series

$$\sum_{n=-\infty}^{\infty} c_n(\tau, t)\xi^n; \qquad c_n(\tau, t) = c_n^{(1)}(\tau) + c_n^{(2)}(\tau)e^{-int}$$

$(c_0(\tau, t) = c_0^{(1)}(\tau))$—functions continuous jointly in the arguments τ and t $(0 \leqslant \tau \leqslant 1, \ 0 \leqslant t \leqslant 2\pi)$, and the series

$$\sum_{n=-\infty}^{\infty} |c_n(\tau, t)|$$

converges uniformly in the rectangle $R = \{\tau, t : 0 \leqslant \tau \leqslant 1, \ 0 \leqslant t \leqslant 2\pi\}$.

Theorem. Let the function $F(\tau, t, \xi) \in \delta$. Then, for the function $f(w, z)$ representable in the form of a Temlyakov integral of the first kind (1) with density $F(\tau, t, \xi)$, the following assertions hold:

1°. At the points of the skeleton $\mathfrak{A}$, as the point (w, z) tends to a point of the skeleton $(w_0, 0) \in \mathfrak{A}$ along paths lying in the nonanalytic hyperplane

$$|z| = k(|w| - c^{-1}),$$

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the formula holds

$$f(w_0,0)=\lim_{(w,z)\to(w_0,0)} f(w,z)=\sum_{n=-\infty}^{\infty}\left[A_{ne}^{in\arg w_0}+B_{ne}^{in\arg 0}\right]^*, \tag{2}$$

where

$$A_n=\lambda_k\,\frac{1}{\pi}\int_0^1 c_n^{(1)}(\tau)\,d\tau;\qquad \lambda_k=\begin{cases}\pi-\varphi_k, & \text{for } n=0,1,2,\dots,\\ -\varphi_k, & \text{for } n=-1,-2,\dots,\end{cases}$$

$$B_n=-\frac{1}{n}\sin(n\varphi_k)\,\frac{1}{\pi}\int_0^1 c_n^{(2)}(\tau)\,d\tau,$$

where

$$\varphi_k=\begin{cases}\arccos(-d/ck), & \text{if the point } (w,z) \text{ approaches a point}\\ & \text{of the skeleton } \mathfrak{A} \text{ along a path belonging to } D^{=};\\ 0, & \text{if the point } (w,z) \text{ approaches a point}\\ & \text{of the skeleton } \mathfrak{A} \text{ along a path belonging to } D^{+};\\ \pi, & \text{if the point } (w,z) \text{ approaches a point}\\ & \text{of the skeleton } \mathfrak{A} \text{ along a path belonging to } D^{-}.\end{cases}$$

The series (2) converges absolutely and uniformly at the points of the skeleton $\mathfrak{A}$.

2°. At the points of the skeleton $\mathfrak{B}$, as the point (w,z) tends to a point of the skeleton $(0,z_0)\in\mathfrak{B}$ along paths lying in the nonanalytic hyperplane

$$|z|=k|w|+d^{-1},$$

the formula holds

$$f(0,z_0)=\lim_{(w,z)\to(0,z_0)} f(w,z)=\sum_{n=-\infty}^{\infty}\left[A_n^* e^{in\arg 0}+B_n^* e^{in\arg z_0}\right], \tag{3}$$

where

$$A_n^*=-\frac{1}{n}\sin(n\varphi_k^*)\cdot\frac{1}{\pi}\int_0^1 c_n^{(1)}(\tau)\,d\tau;$$

$$B_n^*=\lambda_k^*\,\frac{1}{\pi}\int_0^1 c_n^{(2)}(\tau)\,d\tau;\qquad \lambda_k^*=\begin{cases}\pi-\varphi_k^*, & \text{for } n=0,1,2,\dots,\\ -\varphi_k^*, & \text{for } n=-1,-2,\dots,\end{cases}$$

* In formulas (2) and (3), by the argument of zero we shall understand the angle of inclination of the nonanal

where

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$$\varphi_k^* = \begin{cases} \arccos(-dk/c), & \text{if the point } (w,z) \text{ tends to a point of the skeleton } \mathfrak{B} \\ & \text{along a path belonging to } D^-; \\ 0, & \text{if the point } (w,z) \text{ tends to a point of the skeleton } \mathfrak{B} \\ & \text{along a path belonging to } D^+; \\ \pi, & \text{if the point } (w,z) \text{ tends to a point of the skeleton } \mathfrak{B} \\ & \text{along a path belonging to } D^\pm. \end{cases}$$

The series (3) converges absolutely and uniformly at the points of the skeleton $\mathfrak{B}$.

The proof is based on formula (5^{-7}), by means of which, in the space C^2 (for $|w| \neq 0$ and $|z| \neq 0$), the function $f(w,z)$, representable by a Temlyakov integral of the first kind, is given:

$$f(w,z) = (2\pi)^{-1} \int_0^1 \left[\int_{\psi+\varphi}^{2\pi-\varphi+\psi} \Phi(\tau,t,u)\,dt + \int_{\psi-\varphi}^{\psi+\varphi} \Psi(\tau,t,u)\,dt \right] d\tau,$$

where

$$\Phi(\tau,t,u) = (2\pi i)^{-1} \int_{|\zeta|=1} F(\tau,t,\zeta)(\zeta-u)^{-1}\,d\zeta$$

for $|u| < 1$, and the function $\Psi(\tau,t,u)$ is defined by the same formula, but for $|u| > 1$;

$$\psi(w,z) = \arg w - \arg z;$$

$$\varphi(|w|,|z|) = \begin{cases} \arccos a(|w|,|z|), & \text{if } |a(|w|,|z|)| \leq 1, \\ 0, & \text{if } a(|w|,|z|) > 1, \\ \pi, & \text{if } a(|w|,|z|) < -1, \end{cases}$$

$$a(|w|,|z|) = (1 - c^2|w|^2 - d^2|z|^2)(2cd|w||z|)^{-1}.$$

Remark 1. From formula (2) (from formula (3)) it follows that if the point (w,z) tends to the skeleton point $(w_0,0) \in \mathfrak{A}$ (to the point $(0,z_0) \in \mathfrak{B}$) along paths lying in the nonanalytic hyperplane $|z| = k(|w| - c^{-1})$ (in the nonanalytic hyperplane $|z| = k|w| + d^{-1}$) from inside the domain D^-, then the limit of the function $f(w,z)$ (the function $f(w,z)$ representable in the form of a Temlyakov

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integral of the first kind), expressed by formula (2) (formula (3)), remains constant, but its value depends on the angular coefficient of the nonanalytic hyperplane k. Assigning different values to k, i.e. approaching the skeleton points of the domain D of type A along paths lying in different nonanalytic hyperplanes belonging to the domain D^-, we shall obtain different limits for the function $f(w,z)$. This means that, under an arbitrary approach of w and z to the skeleton points of the domain D of type A, the function $f(w,z)$ has no limit. Thus, from what has been said it follows that the Temlyakov integral of the first kind, taken over the boundary of a domain D of type A, has no limit at the skeleton points of the domain D of type A along paths lying in the domain of its nonanalyticity, i.e. from inside the domain D^-, since its limiting values depend on the angular coefficient of the nonanalytic hyperplane containing these paths.

Remark 2. From formula (2) (from formula (3)) it follows that if the approach of the point (w,z) to the skeleton point $(w_0,0) \in \mathfrak{A}$ (to the point $(0,z_0) \in \mathfrak{B}$) takes place along paths lying in nonanalytic hyperplanes $|z| = k(|w| - c^{-1})$ (in nonanalytic hyperplanes $|z| = k|w| + d^{-1}$) from inside the domain D^+ (from inside the domains $D^{\mp}$ and $D^{\pm}$), then the limiting values of the function $f(w,z)$ ($f(w,z)$ is a function representable by a Temlyakov integral of the first kind) at the skeleton points of a domain D of type A are determined, respectively, by the following uniformly and absolutely convergent series (4) and (5); (6) and (7):

$$f_+^+(w_0,0) = \lim_{\substack{(w,z)\to(w_0,0)\\(w,z)\in D^+}} f_+^+(w,z) = \sum_{n=0}^{\infty} a_n^{(1)} e^{in\arg w_0}; \tag{4}$$

$$f^+(0,z_0) = \lim_{\substack{(w,z)\to(0,z_0)\\(w,z)\in D^+}} f^+(w,z) = \sum_{n=0}^{\infty} a_n^{(2)} e^{in\arg z_0}; \tag{5}$$

$$f^-(w_0,0) = \lim_{\substack{(w,z)\to(w_0,0)\\(w,z)\in D^-}} f^-(w,z) = -\sum_{n=1}^{\infty} a_{-n}^{(1)} e^{-in\arg w_0}; \tag{6}$$

$$f^{\pm}(0,z_0) = \lim_{\substack{(w,z)\to(0,z_0)\\(w,z)\in D^{\pm}}} f^{\pm}(w,z) = -\sum_{n=1}^{\infty} a_{-n}^{(2)} e^{-in\arg z_0}, \tag{7}$$

where

$$a_n^{(i)} = \int_0^1 c_n^{(i)}(t)\,dt; \qquad a_{-n}^{(i)} = \int_0^1 c_{-n}^{(i)}(t)\,dt \qquad (i=1,2).$$

It follows from formulas (4) and (5); (6) and (7) that the limiting values of the Temlyakov integral of the first kind at points of the skeleton of a domain D of

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type A, along paths lying in its domain of analyticity, do not depend on the angular coefficient of the nonanalytic hyperplane containing these paths.

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Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

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REFERENCES

[1] A. A. Temlyakov, DAN, **120**, No. 5 (1958).
[2] A. A. Temlyakov, Izv. AN SSSR, ser. matem., **21**, 89 (1957).
[3] A. A. Temlyakov, Uch. zap. Mosk. obl. ped. inst. im. N. K. Krupskaya, **21**, 7 (1954).
[4] B. A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables*, Moscow, 1962.
[5] L. A. Aizenberg, Uch. zap. Mosk. obl. ped. inst. im. N. K. Krupskaya, **96**, 15 (1960).
[6] L. A. Aizenberg, DAN, **120**, No. 5 (1958).
[7] G. L. Lukankin, Uch. zap. Mosk. obl. ped. inst. im. N. K. Krupskaya, **137**, 79 (1964).

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