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Abstract

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ON A PROPERTY OF ALTERNATIVE QUASIGROUPS WITH AN INVARIANT FINITE MEASURE

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Let $\mathfrak{G}$ be a quasigroup ([1]), i.e., a set with an everywhere-defined internal law of composition, for which the left translations γ_σ and the right translations δ_σ, for any $\sigma \in \mathfrak{G}$, are one-to-one mappings of $\mathfrak{G}$ onto itself. In what follows we shall always assume that the quasigroup $\mathfrak{G}$ admits a finite invariant measure, by means of which invariant integration is established on $\mathfrak{G}$.

Thus, to each set $\mathfrak{M} = \{\sigma\}$ of elements $\sigma \in \mathfrak{G}$, belonging to a sufficiently broad class of measurable sets, one can assign a number

$$\int_{(\mathfrak{M})} d\sigma \geqslant 0 \qquad \left(\int_{(\mathfrak{G})} d\sigma = 1\right), \tag{1}$$

and moreover

$$\int_{(\tau\mathfrak{M})} d\sigma = \int_{(\mathfrak{M}\tau)} d\sigma = \int_{(\mathfrak{M})} d\sigma$$

for any $\tau \in \mathfrak{G}$.

The condition of invariance of integration on the quasigroup is expressed by the formula

$$\int_{(\mathfrak{G})} f(\tau\sigma)\, d\sigma = \int_{(\mathfrak{G})} f(\sigma\tau)\, d\sigma = \int_{(\mathfrak{G})} f(\sigma)\, d\sigma. \tag{2}$$

In particular, any bicompact topological group may be taken as $\mathfrak{G}$, since on such a group one can always establish a finite invariant Haar measure and introduce invariant integration ([2,3]).

If $\mathfrak{G}$ is a compact Lie group of dimension n, then

$$\int_{(\mathfrak{M})} d\sigma = \int_{(\mathfrak{M})} \varphi^n \qquad (\varphi^n = \lambda \cdot \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_n).$$

Here $\omega_1, \omega_2, \dots, \omega_n$ are linearly independent left-invariant Maurer–Cartan differential forms; λ is a constant determined by the condition

$$\int_{(\mathfrak{G})} \varphi^n = 1.$$

(By virtue of the compactness of the Lie group under consideration, φ^n is both left-invariant and right-invariant ([4]).)

The condition of invariance of integration

$$\int_{(\mathfrak{G})} f \circ \Phi_\tau \varphi^n = \int_{(\mathfrak{G})} f \circ \Phi_\tau^* \varphi^n = \int_{(\mathfrak{G})} f \varphi^n$$

where Φ_τ is the left (Φ_τ^* the right) translation of the given compact group $\mathfrak{L}$, determined by its element τ, can naturally be written in the form (2).

When considering a finite quasigroup consisting of n elements,

$$\int\limits_{\mathfrak{M}} d\sigma = \frac{m}{n}, \tag{3}$$

where m is the number of elements of the set $\mathfrak{M}$.

A quasigroup is called **alternative** if every one of its subquasigroups generated by any two elements is associative, i.e., is a group. An example of an alternative quasigroup that is not a group is the multiplicative quasigroup of the Cayley algebra, as well as the finite multiplicative subquasigroup of this algebra consisting of 16 elements: $\pm 1,\ \pm i,\ \pm j,\ \pm k,\ \pm e,\ \pm ie,\ \pm je,\ \pm ke.$

Theorem. *Let $\mathfrak{G}$ be an alternative quasigroup with a finite invariant measure. Then, whatever the elements $\tau_1, \tau_2, \dots, \tau_k \in \mathfrak{G}$ and whatever the subset $\mathfrak{M} \subseteq \mathfrak{G}$ satisfying the condition*

$$\int\limits_{\mathfrak{M}} d\sigma > \frac{k-1}{k}, \tag{4}$$

there exists an element $\sigma \in \mathfrak{G}$ ($\sigma' \in \mathfrak{G}$) such that the elements

$$\sigma\tau_1, \dots, \sigma\tau_k \in \mathfrak{M} \qquad (\tau_1\sigma', \dots, \tau_k\sigma' \in \mathfrak{M}). \tag{5}$$

Fig. 1

Figure 1: Fig. 1

Fig. 1

Thus, for example, for the group of the two-dimensional torus T^2 (Fig. 1) there always exists such a translation (both left and right) under which regions (shaded in the figure) with total area (in the usual metric for the torus) constituting more than $(k-1)/k$ of the area of the whole torus cover all k prescribed points on the torus.

Let us also note that in the case of a finite alternative quasigroup condition (4), in view of (3), takes the form

$$m > \mathcal{E}\left(n\frac{k-1}{k}\right).$$

Proof. Consider the set $\mathfrak{N} = \mathfrak{G} \setminus \mathfrak{M}$ and define the function $\mu(\sigma)$ on $\mathfrak{G}$ by the condition

$$\mu(\sigma) = \begin{cases} 1, & \text{if } \sigma \in \mathfrak{N}, \\ 0, & \text{if } \sigma \notin \mathfrak{N} \ (\sigma \in \mathfrak{M}). \end{cases}$$

Then, by virtue of (2), (1), and (4), we have

$$\int\limits_{\mathfrak{G}} \mu(\sigma\tau_i)\, d\sigma = \int\limits_{\mathfrak{G}} \mu(\tau_i\sigma)\, d\sigma = \int\limits_{\mathfrak{G}} \mu(\sigma)\, d\sigma = \int\limits_{\mathfrak{N}} d\sigma = \int\limits_{\mathfrak{G}} d\sigma - \int\limits_{\mathfrak{M}} d\sigma < \frac{1}{k},$$

and consequently the number

$$g = \sum_{i=1}^{k} \int\limits_{\mathfrak{G}} \mu(\sigma\tau_i)\, d\sigma = \sum_{i=1}^{k} \int\limits_{\mathfrak{G}} \mu(\tau_i\sigma)\, d\sigma = k \int\limits_{\mathfrak{G}} \mu(\sigma)\, d\sigma < 1.$$

We shall now show that if in the quasigroup $\mathfrak{G}$ no element $\sigma(\sigma')$ satisfying condition (5) can be found, then necessarily $g \geq 1$.

Indeed, let for every $\sigma \in \mathfrak{G}$ ($\sigma' \in \mathfrak{G}$) among the numbers $(1, 2, \dots, k)$ there be such an i (i') that

$$\sigma\tau_i \in \mathfrak{N} \qquad (\tau_{i'}\sigma' \in \mathfrak{N}).$$

Then, by virtue of the alternativity of the quasigroup under consideration, we have:

$$\bigcup_{i=1}^{k} \mathfrak{M}\tau_i^{-1} = \mathfrak{G} \left(\bigcup_{i'=1}^{k} \tau_{i'}^{-1}\mathfrak{M} = \mathfrak{G} \right). \tag{6}$$

But

$$\int_{(\mathfrak{G})} \mu(\sigma\tau_i)\, d\sigma = \int_{(\mathfrak{M}\tau_i^{-1})} d\sigma \left(\int_{(\mathfrak{G})} \mu(\tau_{i'}\sigma')\, d\sigma' = \int_{(\tau_{i'}^{-1}\mathfrak{M})} d\sigma' \right),$$

and, consequently,

$$g = \sum_{i=1}^{k} \int_{(\mathfrak{G})} \mu(\sigma\tau_i)\, d\sigma = \sum_{i=1}^{k} \int_{(\mathfrak{M}\tau_i^{-1})} d\sigma \geq \int_{\left(\bigcup_{i=1}^{k} \mathfrak{M}\tau_i^{-1}\right)},$$

$$\left(g = \sum_{i'=1}^{k} \int_{(\mathfrak{G})} \mu(\tau_{i'}\sigma')\, d\sigma' = \sum_{i'=1}^{k} \int_{(\tau_{i'}^{-1}\mathfrak{M})} d\sigma' \geq \int_{\left(\bigcup_{i'=1}^{k} \tau_{i'}^{-1}\mathfrak{M}\right)} d\sigma' \right),$$

and therefore, by (6) and (1), $g \geq 1$, and the theorem is proved.

In conclusion we note that if, in the condition of the theorem proved, requirement (4) is replaced by the requirement

$$\int_{(\mathfrak{M})} d\sigma \geq \frac{k-1}{k},$$

then the assertion of the theorem loses its force. For example, for a compact group $\mathfrak{G}$ consisting of k components $(\mathfrak{G}_0, \mathfrak{G}_1, \dots, \mathfrak{G}_{k-1})$, if one chooses one element $\tau_i \in \mathfrak{G}_{i-1}$ for each i, and $\mathfrak{M} = \mathfrak{G} \setminus \mathfrak{G}_\alpha$, where α is any one of the numbers $(0, 1, 2, \dots, k-1)$. Or, for example, for a connected compact commutative Lie group of dimension n. Such a group reduces to the n-dimensional torus T^n ([4]). Here, say, for $n = 1$, when the group T^n is the group of rotations of the circle, it suffices to take as $\tau_1, \tau_2, \dots, \tau_k$ simply any k points lying at the vertices of a regular k-gon inscribed in this circle, and as $\mathfrak{M}$ any open arc of the circle (even with one endpoint included) subtending the central angle $\theta = 2\pi(k-1)/k$. An analogous example, evidently, is easy to indicate also for any group T^n with $n > 1$.

The examples considered show that the theorem proved above, generally speaking, cannot be strengthened. Nevertheless, the question of whether the corresponding theorem, formulated for spaces of transitive (but not simply transitive) representations of groups with invariant volume, can be strengthened remains open for the present.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.