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Abstract

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MATHEMATICS

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INTEGRATION OF ANALYTIC FUNCTIONALS WITH RESPECT TO SIGN-ALTERNATING DISTRIBUTIONS

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The purpose of this note is to prove the integrability of a certain class of analytic functionals with respect to the sign-alternating distribution P_{2q} corresponding to the equation

$$\partial u/\partial t = (-1)^{q+1}\partial^{2q}u/\partial x^{2q}. \tag{1}$$

The distribution P_{2q} is defined in the space of continuous functions $x(t)$, $x(0) = 0$, and is a generalization of Wiener measure ([1]) associated with the heat equation.

Let $C_0[0,1]$ denote the space of functions $x(t)$ continuous on $[0,1]$, with $x(0) = 0$. On all quasi-intervals $K_n(t_k, a_k, b_k)$ of the space $C_0[0,1]$ we define a consistent system of distributions, defining the weight $P_{2q}[K_n]$ of the quasi-interval $K_n(t_k, a_k, b_k)$ by the formula ([2])

$$P_{2q}[K_n] = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{k=0}^{n-1} G_{2q}(t_{k+1} - t_k,\ x_{k+1} - x_k)\, dx_{k+1},$$

where

$$G_{2q}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[-t\xi^{2q} + ix\xi]\, d\xi$$

is the Green function of equation (1).

Suppose that on the space $C_0[0,1]$ a functional $F[x(t)]$ is given. Denote by $F[x_1, \dots, x_n]$ the value of the functional $F[x(t)]$ on the stepwise curve $x^s(t) = x_k$ for $t_k \le t < t_{k+1}$ $(k = 0, \dots, n-1)$, corresponding to the partition $0 < t_1 < \cdots < t_n \le 1$.

We shall say that there exists a continual integral

$$I = \int_{C_0} F[x(t)]\, dP_{2q}[x(t)]$$

of the functional $F[x(t)]$ with respect to the distribution P_{2q} over the whole space $C_0[0,1]$, if there exists a limit $I = \lim_{\lambda \to 0} I_n$, where $\lambda = \max_k (t_{k+1} - t_k)$, of the n-fold integral

$$I_n = \int_{-\infty}^{+\infty} \cdots \int F[x_1, \dots, x_n] \prod_{k=0}^{n-1} G_{2q}(t_{k+1} - t_k,\ x_{k+1} - x_k)\, dx_{k+1},$$

independent of the partition of the interval $[0,1]$ by the points t_k $(k = 1, \dots, n)$.

A functional $F[x(t)]$, defined on $C_0[0,1]$, will be called analytic if it is representable in the form

$$F[x(t)] = \sum_{p=0}^{\infty} \frac{1}{p!} F_p[x(t)], \tag{2}$$

where

$$F_p[x(t)] = \int_0^1 \cdots \int_0^1 h_p(t_1, \dots, t_p)\, dx(t_1) \cdots dx(t_p),$$

the functions $h_p(t_1, \dots, t_p)$ being assumed to depend symmetrically on their arguments.

Moreover, in order that the integral in the definition of $F_p[x(t)]$ have meaning for an arbitrary continuous function $x(t) \in C_0[0,1]$, and also for a step curve $x^s(t)$, we require that the function $h_p(t_1, \dots, t_p)$ and all its derivatives up to order p, taken with respect to each argument no more than once, be continuous and equal to zero if at least one variable $t_j = 1$ $(j = 1, \dots, p)$ identically with respect to the remaining variables. In this case, by definition, we set

$$F_p[x(t)] = (-1)^p \int_0^1 \cdots \int_0^1 \frac{\partial^p h_p(t_1, \dots, t_p)}{\partial t_1 \cdots \partial t_p} x(t_1) \cdots x(t_p)\, dt_1 \cdots dt_p.$$

Theorem*. The continual integral I of the analytic functional $F[x(t)]$ with respect to the distribution P_{2q} on the space $C_0[0,1]$ exists if, for $p = 2ql$ $(l = 1, 2, \dots)$, the functions $h_p(t_1, \dots, t_p)$ satisfy the inequalities

$$\left| h_{2ql}(s_1, \dots, s_1, s_2, \dots, s_2, \dots, s_l, \dots, s_l) \right| < c^l, \tag{3}$$

where the constant $c > 0$ does not depend on l, and in the arguments of the function $h_{2ql}(s_1, \dots, s_1, s_2, \dots, s_2, \dots, s_l, \dots, s_l)$ there are groups of $2q$ identical terms, $0 \leq s_i \leq 1$ $(i = 1, \dots, l)$. Moreover, the equality holds

$$I = \sum_{l=0}^{\infty} \frac{(-1)^{l(q+1)}}{l!} \int_0^1 \cdots \int_0^1 h_{2ql}(s_1, \dots, s_1, \dots, s_l, \dots, s_l)\, ds_1 \cdots ds_l. \tag{4}$$

Proof. The integral of the functional $F_p[x(t)]$, by definition, is equal to the limit, as $\max_k \Delta t_{k+1} \to 0$ $(\Delta t_{k+1} = t_{k+1} - t_k)$, of the $2n$-fold integral

$$I_n^p = \frac{1}{(2\pi)^n} \int \cdots \int_{-\infty}^{+\infty} F_p[x_1, \dots, x_n] \times$$

$$\times \exp\left[-\sum_{k=0}^{n-1} \xi_{k+1}^{2q} \Delta t_{k+1} + i \sum_{k=0}^{n-1} (x_{k+1} - x_k)\xi_{k+1}\right] dx_1 \cdots dx_n\, d\xi_1 \cdots d\xi_n.$$

Here $F_p[x_1, \dots, x_n]$ is the value of the functional $F_p[x^s(t)]$ on the step curve $x^s(t)$. By the definition of the functional $F_p[x(t)]$ it is equal to

$$F_p[x_1, \dots, x_n] = (-1)^p \sum_{0<k_1,\dots,k_p \leq n} \Delta^p h_p(t_{k_1}, \dots, t_{k_p})\, x_{k_1} \cdots x_{k_p} =$$

$$= \sum_{0 \leq k_1,\dots,k_p \leq n-1} h_p(t_{k_1}, \dots, t_{k_p})(x_{k_1+1} - x_{k_1}) \cdots (x_{k_p+1} - x_{k_p}).$$

Let us note that $F_p[x_1, \dots, x_n]$ is a homogeneous polynomial of degree p in $x_1, \dots, x_n$. Making in I_n^p the triangular change of variables $x_{k+1} = z_1 \Delta t_1 + \cdots + z_{k+1} \Delta t_{k+1}$, $\xi_{k+1} \Delta t_{k+1} = \eta_{k+1}$ $(k = 0, \dots, n-1)$, and introducing the notation $H_p[z_1, \dots, z_n] = F_p[z_1 \Delta t_1, \dots, z_1 \Delta t_1 + \cdots + z_n \Delta t_n]$, we obtain

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* E. V. Maikov in [3] gave another definition of integrals with respect to generalized measures for functionals of a more general type, which he called τ-analytic, without carrying out the computation of the integrals for concrete measures.

$$I_n^p = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \cdots \int H_p[z_1, \dots, z_n] \times$$

$$\times \exp\left[-\sum_{k=0}^{n-1} \frac{\eta_{k+1}^{2q}}{(\Delta t_{k+1})^{2q}} \Delta t_{k+1} + i \sum_{k=0}^{n-1} z_{k+1} \eta_{k+1}\right] dz_1 \dots dz_n\, d\eta_1 \dots d\eta_n.$$

Since $H_p[z_1, \dots, z_n]$ is a homogeneous polynomial of degree p in $z_1, \dots, z_n$, we have $I_n^p = 0$ if p is not divisible by $2q$. If $p = 2ql$, then for I_n^{2ql} we obtain the expression

$$I_n^{2ql} = \frac{(-1)^{l(q+1)}}{l!} \times$$

$$\times \sum_{l_1+\dots+l_n=l} \frac{l!}{l_1! \cdots l_n!} \frac{\partial^{2ql} H_{2ql}[z_1, \dots, z_n] (\Delta t_1)^{l_1} \cdots (\Delta t_n)^{l_n}}{\partial z_1^{2ql_1} \dots \partial z_n^{2ql_n} (\Delta t_1)^{2ql_1} \cdots (\Delta t_n)^{2ql_n}} \Bigg|_{z_1=\dots=z_n=0}.$$

By definition,

$$H_{2ql}[z_1, \dots, z_n] = \sum_{0 \le k_1, \dots, k_{2ql} \le n-1} h_{2ql}(t_{k_1}, \dots, t_{k_{2ql}}) z_{k_1} \Delta t_{k_1} \cdots z_{k_{2ql}} \Delta t_{k_{2ql}},$$

where $h_p(t_1, \dots, t_p)$ is a symmetric function of its arguments, so that

$$\frac{\partial^{2ql} H[z_1, \dots, z_n]}{\partial z_1^{2ql_1} \dots \partial z_n^{2ql_n}} \Bigg|_{z_1=\dots=z_n=0} = (2ql)!\, h_{2ql}(t_1, \dots, t_1, \dots, t_n, \dots, t_n) (\Delta t_1)^{2ql_1} \cdots (\Delta t_n)^{2ql_n},$$

where in the arguments of the function $h_{2ql}(t_1, \dots, t_1, \dots, t_n, \dots, t_n)$ there are groups of $2ql_1, \dots, 2ql_n$ identical $t_1, \dots, t_n$, respectively. Consequently, for I_n^{2ql} we obtain

$$I_n^{2ql} \frac{l!}{(-1)^{l(q+1)} (2ql)!} = \sum_{l_1+\dots+l_n=l} \frac{l!}{l_1! \cdots l_n!} h_{2ql}(t_1, \dots, t_1, \dots, t_n, \dots, t_n) \times$$

$$\times (\Delta t_1)^{l_1} \cdots (\Delta t_n)^{l_n} =$$

$$= \frac{(-1)^{l(q+1)} (2ql)!}{l!} \sum_{0 < j_1, \dots, j_l \le n} h_{2ql}(t_{j_1}, \dots, t_{j_1}, \dots, t_{j_l}, \dots, t_{j_l}) \Delta t_{j_1} \cdots \Delta t_{j_l}. \quad (5)$$

The approximate value I_n of the analytic functional (2) for any fixed partition of the interval $[0, 1]$ is defined as the limit

$$I_n = \lim_{M \to \infty} \sum_{l=0}^{M} \frac{1}{(2ql)!} I_n^{2ql}. \quad (6)$$

This limit exists under the assumptions made in (3), since for any $\varepsilon > 0$ and any partition of the interval $[0,1]$ there exists a number M such that for all $M' > M$, in view of formula (5),

$$\left|\sum_{l=M'}^{\infty} \frac{1}{(2ql)!} I_n^{2ql}\right| \leq \sum_{l=M'}^{\infty} \frac{c^l}{l!} < \frac{\varepsilon}{2}.$$

On the other hand, since the functions $h_p(t_1, \dots, t_p)$ are continuous by assumption, one can choose $\delta = \delta(\varepsilon)$ such that for all partitions for which $\lambda = \max_k \Delta t_k < \delta$, the inequality

$$\left|\sum_{l=0}^{M} \frac{1}{(2ql)!} I_n^{2ql} - \sum_{l=0}^{M} \frac{1}{l!} \int_0^1 \cdots \int_0^1 h_{2ql}(s_1, \dots, s_1, \dots, s_l, \dots, s_l)\, ds_1 \cdots ds_l\right| < \frac{\varepsilon}{2}$$

holds, since expression (5) is an integral sum for the integral of the continuous function $h_{2ql}(s_1, \dots, s_1, \dots, s_l, \dots, s_l)$ over the l-dimensional cube. The theorem is proved.

We note that for the integral I_n (6) the formula holds

$$I_n = \sum_{l=0}^{\infty} \frac{(-1)^{l(q+1)}}{l!} \times$$

$$\times \sum_{l_1+\dots+l_n=l} \frac{l!}{l_1! \cdots l_n!} \left. \frac{\dfrac{\partial^{2ql} H_{2ql}[z_1, \dots, z_n](\Delta t_1)^{l_1} \cdots (\Delta t_n)^{l_n}}{\partial z_1^{2ql_1} \cdots \partial z_n^{2ql_n} (\Delta t_1)^{2ql_1} \cdots (\Delta t_n)^{2ql_n}}}{} \right|_{z_1=\dots=z_n=0},$$

which is conveniently written in the form

$$I_n = \exp\left\{(-1)^{q+1} \sum_{k=0}^{n-1} \frac{\partial^{2q}}{\partial z_{k+1}^{2q} (\Delta t_{k+1})^{2q}} \Delta t_{k+1}\right\} H[z_1, \dots, z_n]\Big|_{z_1=\dots=z_n=0}.$$

For the analytic functional $F[x(t)]$, by what has been proved, there exists the limit of I_n as $\max_k \Delta t_k \to 0$. We make this formal passage to the limit in the last expression for I_n. We obtain the formula

$$I = \exp\left\{(-1)^{q+1} \int_0^1 \frac{\delta^{2q}}{[\delta z(t)]^{2q}}\, dt\right\} F\left[\int_0^t z(s)\, ds\right]\Bigg|_{z(s)\equiv 0}, \tag{7}$$

which reduces continual integration of a functional to the application to it, at $z(s) \equiv 0$, of an operator in variational derivatives. We note that expression (4) can also be obtained formally from (7), using the fact that for an analytic functional

$$\left\{ \delta^p F\left[\int_0^t z(s)\,ds\right] / \delta z(t_1) \cdots \delta z(t_p) \right\}\Big|_{z(s)\equiv 0} = h_p(t_1, \dots, t_p).$$

As an example of the application of the theorem and formula (7), let us compute the characteristic functional $\Phi[y(t)]$ of the distribution P_{2q}.

The characteristic functional $\Phi[y(t)]$ of the distribution P_{2q} is the continual integral over the entire space $C_0[0, 1]$ of the functional

$$E[x(t)] = \exp\left[i \int_0^1 x(t)y(t)\,dt \right].$$

Since the functional $E[x(t)]$ satisfies the conditions of the theorem for every continuous function $y(t)$, $y(1) = 0$, the characteristic functional $\Phi[y(t)]$ exists. Formula (7) in fact makes it possible to compute $\Phi[y(t)]$. Indeed, by formula (7) we have

$$\Phi[y(t)] = \exp\left\{ (-1)^{q+1} \int_0^1 \frac{\delta^{2q}}{[\delta z(t)]^{2q}}\,dt \right\} \exp\left\{ i \int_0^1 y(t) \left[\int_0^t z(s)\,ds \right] dt \right\}\Big|_{z(s)\equiv 0}.$$

Changing the order of integration in the last exponent and noting that

$$\frac{\delta^p}{[\delta z(t)]^p} \exp\left\{ i \int_0^1 z(t) \left[\int_t^1 y(s)\,ds \right] dt \right\}\Big|_{z(s)\equiv 0} = \left[i \int_t^1 y(s)\,ds \right]^p,$$

we finally obtain for $\Phi[y(t)]$ the expression

$$\Phi[y(t)] = \exp\left\{ -\int_0^1 \left[\int_t^1 y(s)\,ds \right]^{2q} dt \right\}.$$

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Note: Figure translations are in progress. See original paper for figures.