SOME CASES OF UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM FOR STRINGS WITH A BOUNDARY CONDITION AT A SINGULAR END

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Abstract

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SOME CASES OF UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM FOR STRINGS WITH A BOUNDARY CONDITION AT A SINGULAR END

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1. As in ([2]), we assign a nondecreasing function $M(x)$ $(-\infty < x < b \leq +\infty;\ M(x-0) = M(x))$ to the class $\mathfrak{M}$ if $M(x) \in L^1(-\infty, c)$, where $-\infty < c < b$. With the function $M(x)$ $(-\infty < x < b)$ we associate a string S, extending along the x-axis from the point $x = -\infty$ to the point $x = b$, the mass of each interval $[\alpha, \beta)$ of which is equal to $M(\beta) - M(\alpha)$. When the function $M(x)$ $(-\infty < x < b)$ belongs to the class $\mathfrak{M}$, we shall say that the string S associated with it belongs to the class $\mathfrak{M}_S$. The left end $x = -\infty$ of the string S is called **regular** if the set of values of the function $M(x)$ and the set of its points of increase are bounded below. Otherwise the end $x = -\infty$ is called **singular**. Analogous definitions are adopted for the right end $x = b$.

As in ([2]), we consider the differential system (boundary-value problem)

$$-\frac{d}{dM(x)}\, y^-(x) - \lambda y(x) = 0 \quad (-\infty < x < b); \qquad \lim_{x\downarrow-\infty} y(x) = 1, \tag{1}$$

where $y^-(x)$ denotes the left derivative of the function $y(x)$. Already in ([1]) (a special case of Theorem 3) we established that, for any complex λ, the system (1) has a unique solution $\Phi(x, \lambda)$ in the class of absolutely continuous functions having at every point $x \in (-\infty, b)$ a left derivative which, in turn, is M-absolutely continuous. The function $\Phi(x, \lambda)$, for any fixed x, is an entire function of λ and, as we have recently established, its growth is no greater than that of minimal type of order one.

Let $L^2_M(-\infty, b)$ be the Hilbert space of all complex-valued M-measurable functions $f(x)$ $(-\infty < x < b)$ whose square is M-summable on $(-\infty, b)$. By

$\overset{\circ}{L}{}^2_M(-\infty, b)$ we denote its subspace consisting of functions equal to zero in neighborhoods of the singular ends of the interval $(-\infty, b)$, the neighborhoods being individual for each function. A nondecreasing function $\tau(\lambda)$ $(-\infty < \lambda < +\infty)$ is called a **spectral function** of the string S (of the boundary-value problem (1)) if the mapping $U : f \mapsto F$, where $f \in \overset{\circ}{L}{}^2_M(-\infty, b)$, and

$$F(\lambda) = \int_{-\infty}^{b} f(x)\Phi(x, \lambda)\, dM(x)$$

isometrically maps $\overset{\circ}{L}{}^2_M(-\infty, b)$ into $L^2_\tau(-\infty, +\infty)$. The spectral function $\tau(\lambda)$ is called **orthogonal** if the mapping U maps $L^2_M(-\infty, b)$ into a dense part of $L^2_\tau(-\infty, +\infty)$. The spectral function $\tau(\lambda)$ is called **positive** if the τ-measure of the negative half-axis $(-\infty, 0)$ is equal to zero.

As we have already indicated in ([2]), our Theorem 3 from ([1]) contains, as a special case $(Q(x) \equiv 0)$, the assertion that the string S

(the boundary-value problem (1)) under the condition that $S \in \mathfrak{M}_s$ (and, consequently, independently of the behavior of the function $M(x)$ in a neighborhood of the point $x = b$) has at least one orthogonal spectral function even in the case when the left endpoint $x = -\infty$ is singular and the limit-point case holds there. Among the orthogonal spectral functions of the string S there are positive ones.

2. In (2) the set T of all functions $\tau(\lambda)$ that are spectral functions of strings S belonging to the class $\mathfrak{M}_s$ was studied. In (2) a number of sufficient conditions for membership in the set T were given; here, in Theorems 2 and 3, two more such conditions will be given.

The inverse problem referred to in the title of the article consists in seeking, from a given function $\tau(\lambda) \in \mathrm{T}$, a string S for which $\tau(\lambda)$ is a spectral function. This problem has a "natural" nonuniqueness.

We shall call two strings S_1 and S_2 from $\mathfrak{M}_s$ equivalent if, for the functions $M_1(x)$ $(-\infty < x < b_1)$ and $M_2(x)$ $(-\infty < x < b_2)$ associated with them, there exists a real constant h such that a point x is a point of increase of the function $M_1(x)$ if and only if it is a point of increase of the function

$$M_2^{(h)}(x) = M_2(x + h)$$

and at these points the equality

$$M_1(x) = M_2(x + h)$$

holds. We shall call a string $S_1 \in \mathfrak{M}_s$ the left part of a string $S_2 \in \mathfrak{M}_s$ if the first of them is obtained from the second by discarding, from the latter, a piece on the right-hand side, i.e., the functions $M_1(x)$ $(-\infty < x < b_1)$ and $M_2(x)$ $(-\infty < x < b_2)$ associated with them are such that $b_1 < b_2$, and for every

$x < b_1$ the equality $M_1(x) = M_2(x)$ holds. In this case we shall call the string S_1 a proper left part of the string S_2 if the M_2-measure of the interval $[b_1, b_2)$ is different from zero. It is clear that equivalent strings belonging to the class $\mathfrak{M}_s$ have the same sets of spectral functions, and if a string S_1 is the left part of a string $S_2 \in \mathfrak{M}_s$, then every spectral function of the string S_2 is a spectral function of the string S_1. This is the "natural" nonuniqueness of the inverse problem.

In this connection we shall say that, for some function $\tau(\lambda) \in \mathrm{T}$, the inverse problem is uniquely solvable if, of any two strings S_1 and S_2 from $\mathfrak{M}_s$ whose spectral function is $\tau(\lambda)$, one is equivalent to a left part of the other. In this case, obviously, there is a string $\hat{S}$ such that any string S for which the function $\tau(\lambda)$ is spectral is equivalent to a left part of the string $\hat{S}$. We shall call the string $\hat{S}$ the maximal solution of the inverse problem for the function $\tau(\lambda)$. The maximal solution is determined uniquely up to equivalence. If, for a function $\tau(\lambda) \in \mathrm{T}$, the inverse problem is uniquely solvable and S is its maximal solution, then, as is easy to see, $\tau(\lambda)$ cannot be an orthogonal spectral function of a string from $\mathfrak{M}_s$ not equivalent to the string $\hat{S}$; at the same time it may happen that $\tau(\lambda)$ is not an orthogonal spectral function of any string. Nevertheless, in the case when the function $\tau(\lambda) \in \mathrm{T}$ has no points of increase on the interval $(-\infty, 0)$, it is always an orthogonal spectral function of at least one string from $\mathfrak{M}_s$.

Theorem 1. *If a nondecreasing function $\tau(\lambda)$ $(-\infty < \lambda < +\infty)$ has no points of increase on the half-line $-\infty < \lambda < 0$ and is majorized on the half-line $0 \leqslant \lambda < +\infty$ by some polynomial in λ, then the inverse problem* * *for the function $\tau(\lambda)$ is uniquely solvable.*

Theorem 2. *Let $\tau(\lambda)$ $(-\infty < \lambda < +\infty)$ be a nondecreasing function whose points of increase (jump points) are only positive numbers*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

such that

$$\sum \lambda_j^{-1} < \infty,$$

and the jumps

* In (2) we indicated that a function $\tau(\lambda)$ possessing these properties belongs to T.

$$\tau(\lambda_j + 0) - \tau(\lambda_j - 0) = \nu_j \quad (j = 1, 2, 3, \ldots)$$

of the function $\tau(\lambda)$ at these points are such that, for some $k \geqslant 1$,

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{k+1} |D'(\lambda_j)|^2 \nu_i} < \infty,$$

where $D(z)$ is the entire function defined by the equality

$$D(z) = \prod_{j=1}^{\infty}(1 - z/\lambda_j).$$

Then the function $\tau(\lambda)$ belongs to T, and for it the inverse problem is uniquely solvable.

Theorem 3. Let $\tau(\lambda)$ $(-\infty < \lambda < +\infty)$ be a nondecreasing function whose points of increase (points of jumps) are only the numbers

$$\mu_0 = 0 < \mu_1 < \mu_2 < \ldots$$

such that

$$\sum_{j=1}^{\infty} \mu_j^{-1} < \infty,$$

and suppose the jumps

$$\tau(\mu_j + 0) - \tau(\mu_j - 0) = \theta_j \quad (j = 0, 1, 2, \ldots)$$

of the function $\tau(\lambda)$ at these points are such that, for some $k \geqslant 1$,

$$\sum_{j=1}^{\infty} \mu_j^{-k} \left(E'(\mu_j)\right)^{-2} \theta_j^{-1} < \infty,$$

where $E(z)$ is the entire function defined by the equality

$$E(z) = -z \prod_{j=1}^{\infty}(1 - z/\lambda_j).$$

Then the function $\tau(\lambda)$ belongs to T, and for it the inverse problem is uniquely solvable.

The following theorems give sufficient conditions for the function $\tau(\lambda)$ to satisfy the conditions of one of the theorems stated above (and, consequently, for the inverse problem for $\tau(\lambda)$ to be uniquely solvable), in terms of a string S for which it is known in advance that $\tau(\lambda)$ is its orthogonal spectral function. At the same time, these same theorems (in the part asserting necessity) give information about the maximal solution $\hat{S}$ of the inverse problem for the function $\tau(\lambda)$, if it is known that it satisfies the condition of one of Theorems 1, 2, and 3; for in this case the string S, for which $\tau(\lambda)$ is an orthogonal spectral function, is equivalent, as we clarified above, to the string $\hat{S}$.

Theorem 4*. In order that the function $\tau(\lambda)$ satisfy the conditions of Theorem 1, it is necessary and sufficient that it be a positive orthogonal spectral function of a string S such that, for some $\varepsilon > 0$, as $x \downarrow -\infty$ the asymptotic equality

$$M(x) = O(|x|^{-1-\varepsilon})$$

holds.

Theorem 5. In order that the function $\tau(\lambda)$ satisfy, for $k = 1$, the condition of Theorem 2 or Theorem 3, it is necessary and sufficient that it be a positive orthogonal spectral function of some string $S \in \mathfrak{M}_S$ with a regular right end.

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* In Theorems 4, 5, 6, 7, $M(x)$ $(-\infty < x < b)$ is the function associated with the string S.

Theorem 6. In order that the function $\tau(\lambda)$ satisfy the condition of Theorem 2 for $k > 1$, and for $k = 1$ not satisfy it, it is necessary and sufficient that it be the orthogonal spectral function of a string $S \in \mathfrak{M}_s$ such that $b < \infty$, the end $x = b$ is singular $(M(b) = \infty)$, and for some $\varepsilon > 0$ the following asymptotic equality holds as $x \uparrow b$:

$$M(x) = O\big((b - x)^{-1+\varepsilon}\big).$$

Theorem 7. In order that the function $\tau(\lambda)$ satisfy the condition of Theorem 3 for $k > 1$, and for $k = 1$ not satisfy it, it is necessary and sufficient that it be the orthogonal spectral function of a string $S \in \mathfrak{M}_s$ such that $b = +\infty$, the end $x = b = +\infty$ is singular, $M(+\infty) < \infty$, and for some $\varepsilon > 0$ the following asymptotic equality holds as $x \uparrow +\infty$:

$$M(+\infty) - M(x) = O(x^{-1-\varepsilon}).$$

In the proof of Theorems 1, 2, and 3 we used the remarkable theorem of M. G. Krein ((3), Theorem 1), which gives conditions necessary and sufficient for a nondecreasing function $\tau(\lambda)$ to be a positive spectral function of a string with a boundary condition at the regular end, and which asserts the unique solvability of such a problem. In his recent works L. de Branges, apparently unaware of M. G. Krein' s works, generalized this theorem ((4), Theorems XI and XII; (5), Theorem III; (6), Theorem VII). He also generalized ((4), Theorem IV) our Theorem 3 from (1). In connection with this, all the theorems of the present article can be transferred to the scheme of L. de Branges.

Let us note, incidentally, that L. de Branges, in the above-mentioned papers and in a number of his other works, studies Hilbert spaces of entire functions, without knowing that the theory of such spaces had already been developed in the works (7, 8) of M. G. Krein.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.