EFFECTIVE TRANSFORMATION AND ASYMPTOTICS OF ONE CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract

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EFFECTIVE TRANSFORMATION AND ASYMPTOTICS OF ONE CLASS OF NON-LINEAR DIFFERENTIAL EQUATIONS

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Consider the autonomous system*

$$dx/dt = -y + \varepsilon\Pi_1(p,x,y), \qquad dy/dt = x + \varepsilon\Pi_2(p,x,y),$$

$$dp/dt = \varepsilon\Pi_3(p,x,y), \tag{1}$$

where x, y, Π_1, Π_2 are scalars; p, Π_3 are n-dimensional vectors; ε is a small parameter; Π_1, Π_2, Π_3 are functions of p, x, y that are three times continuously differentiable in a domain D, in which it is possible to specify such positive constants M, λ that the inequalities

$$|\Pi_i| \le M, \qquad |\partial\Pi_i/\partial x_i| \le \lambda \quad (i = 1,2,3;\ x_i = p,x,y) \tag{2}$$

are satisfied.

Let $x = \alpha(p),\ y = \beta(p)$ satisfy equations (1). Introduce in (1) the change of variables

$$x = \alpha + A\cos(t-\varphi), \qquad y = \beta + A\sin(t-\varphi), \qquad p = p, \tag{3}$$

where A, φ are new variables, $A \ge 0$; then (1) is transformed to the form

$$dA/dt = \varepsilon AF(p,x,y), \qquad d\varphi/dt = \varepsilon\Phi(p,x,y),$$

$$dp/dt = \varepsilon\Pi_3(p,x,y), \tag{4}$$

where

$$F = \frac{1}{A}\{[\Pi_1(p,x,y) - \Pi_1(p,\alpha,\beta)]\cos(t-\varphi) +$$

$$+[\Pi_2(p,x,y) - \Pi_2(p,\alpha,\beta)]\sin(t-\varphi)\}, \tag{5}$$

$$\Phi = \frac{1}{A}\{[\Pi_1(p,x,y) - \Pi_1(p,\alpha,\beta)]\sin(t-\varphi) -$$

$$-[\Pi_2(p,x,y) - \Pi_2(p,\alpha,\beta)]\cos(t-\varphi)\}.$$

Thanks to the terms $\Pi_1(p,\alpha,\beta)$, $\Pi_2(p,\alpha,\beta)$, the Taylor expansions of (5) in powers of A in a neighborhood of $A=0$ ($x=\alpha,\ y=\beta$) will begin with terms bounded by the constant 3λ, as a result of which, both for small A and for $A \sim 1$ in the domain D, there exist constants M_1, λ_1 bounding, respectively, F, Φ, Π_3 and their derivatives with respect to p, x, y. We shall use this property in averaging system (4).

Consider a system in the standard form for the averaging method ([2])

$$dx/dt = \varepsilon X(t,x), \tag{6}$$

where x, X are n-dimensional vectors, ε is a small parameter. Here the function $X(t,x)$: a) is periodic in t with period T of order unity; b) is continuously differentiable with respect to t in the domain D, in which inequalities analogous to (2) are satisfied for $X(t,x)$.

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* A particular case of (1) is the equations of celestial mechanics in the form ([1]).

Let us write the system of averaged equations

$$d\xi/dt = \varepsilon X_0(\xi) \qquad \left(X_0(\xi) = \frac{1}{T}\int_0^T X(t,\xi)\,dt\right), \tag{7}$$

where $\xi(t)$ is defined and lies in the domain D together with its neighborhood of radius ρ.

We represent the solution of (6) in the form

$$x = \xi + \varepsilon \widetilde{X}(t,\xi) + \varepsilon r(t), \tag{8}$$

where $r(t)$ is a new variable, $x(0) = \xi(0)$, $r(0) = 0$,

$$\widetilde{X}(t,\xi) = \int_0^t [X(t,\xi) - X_0(\xi)]\, dt. \tag{9}$$

Obviously, in the domain D we have

$$|X|,\ |X_0| \leq M, \qquad |\partial X/\partial x| \leq \lambda, \qquad |\partial \widetilde{X}/\partial \xi| \leq \lambda T. \tag{10}$$

Substituting (8) into (6) and taking account of (7), (9), we obtain the equation

$$dr/dt = -\varepsilon X_0(\xi)\partial \widetilde{X}(t,\xi)/\partial \xi + X\{t, \xi + \varepsilon \widetilde{X}(t,\xi) + \varepsilon r\} - X(t,\xi). \tag{11}$$

Introducing the slow time $\tau = \varepsilon t$ and rewriting (11) in the form of an integral equation, we define r by means of Picard successive approximations

$$r = r_0 + (r_1 - r_0) + \dots + (r_n - r_{n-1}) + \dots, \tag{12}$$

where

$$r_0 = 0, \dots, \quad r_n = \int_0^\tau \left[- X_0(\xi)\frac{\partial \widetilde{X}}{\partial \xi} + \frac{X(t, \xi + \varepsilon \widetilde{X} + \varepsilon r_{n-1}) - X(t,\xi)}{\varepsilon} \right] d\tau. \tag{13}$$

With the aid of (10) we obtain the estimates

$$|r_1| \leq 2TM\lambda\tau, \dots, \quad |r_n - r_{n-1}| \leq 2TM\lambda^n\tau^n/n!, \dots \tag{14}$$

Substituting (12) into (8), it is easy to see that the successive approximations for (8) will not leave the domain D if the averaging error $\eta = x - \xi$ does not exceed ρ:

$$|\eta| = \varepsilon|\widetilde{X} + r| \leq T\varepsilon M(2\exp\lambda\tau - 1) \leq \rho. \tag{15}$$

Inequality (15) is satisfied by $\tau \in [0, L]$, where

$$L \leq \frac{1}{\lambda} \ln \frac{1}{2} \left(1 + \frac{\rho}{T\varepsilon M} \right). \tag{16}$$

By analogy with the theorem on the existence of solutions, one can show that for $0 \leq \tau < L$ the series (12) converges uniformly to the solution of (11), which is unique.

Thus, on the interval $0 < t < L/\varepsilon$ the averaging error* does not exceed ρ and consists of two parts: an oscillatory part $\varepsilon\widetilde{X} \sim \varepsilon TM$ and an accumulating part $\varepsilon r \sim 2\varepsilon TM(\exp\lambda\tau - 1)$.

We apply the result obtained to system (4). Introduce the notation: the $(n+2)$-dimensional vector $X = \{AF, \Phi, \Pi_z\}$; the vector $\xi(t)$ is the solution of the averaged equations (4); $X_0(\xi)$ is the vector X averaged over a period; the vector $\widetilde{X}(t, \xi)$ is the operator (9). Obviously, the first components of the vectors X, X_0

* Here, in contrast to (2), the estimates for the averaging error are obtained in explicit form.

are bounded by the quantity AM_1, and the remaining ones by the quantity M_1. Taking this into account, and for $A \leqslant \varepsilon$, from the Taylor expansions of F, Φ, Π_3, one can show that

$$|\widetilde{X}| \leqslant 2\pi AM_1, \qquad |X_0 \partial\widetilde{X}/\partial\xi| \leqslant 2\pi\lambda AM_1. \tag{17}$$

The error of averaging will be $\eta = \varepsilon(\widetilde{X} + r)$, where r satisfies an equation of the form (11).

From the first equation (4) it follows easily that

$$A \leqslant A_0 \exp \varepsilon M_1 t, \tag{18}$$

or

$$t \geqslant \frac{1}{\varepsilon M_1} \ln \frac{A}{A_0}, \tag{19}$$

i.e. the transition from values $A \sim \varepsilon$ to $A \sim 1$ lasts on the interval

$$t \sim \frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}.$$

Using (17), (18), let us estimate the terms of the series (12) on the interval (19):

$$|r_1| \leqslant 4\pi\lambda M_1 A_0 \exp M_1\tau, \dots, \; |r_n - r_{n-1}| \leqslant 4\pi A_0 (\exp M_1)\tau^n \lambda_1^{n+1}/M_1^n, \tag{20}$$

whence, for $\lambda_1/M_1 < 1$, we have

$$\varepsilon|r| \leqslant 4\pi\varepsilon\lambda M_1 A/(1 - \lambda_1/M_1). \tag{21}$$

Thus, depending on the initial conditions, the interval of validity of solutions of the averaged equations (4) can be significantly extended in comparison with theorem (2) on the averaging of system (7).

In addition, in the case of decreasing A, the range of the values of ε under consideration can be extended. Indeed, the right-hand side of (21) contains εA. The smaller A is, the larger the ε that may be prescribed.

Let us describe a method of approximately finding α, β, which is a generalization of the corresponding result of (1). Let Π_1, Π_2, Π_3 be analytic functions of p, x, y. We shall seek α, β in the form of series

$$\alpha = \alpha_0 + \ldots + (\alpha_n - \alpha_{n-1}) + \ldots,$$

$$\beta = \beta_0 + \ldots + (\beta_n - \beta_{n-1}) + \ldots. \tag{22}$$

Here α_n, β_n are partial sums; the general terms $(\alpha_n - \alpha_{n-1})$, $(\beta_n - \beta_{n-1})$ will be denoted respectively by $\varepsilon^{n+1} Q_n$, $\varepsilon^{n+1} R_n$.

We expand the functions Π_1, Π_2 in Taylor series in powers of x, y in a neighborhood of $x = y = 0$ and determine α_0, β_0 from the equations

$$-\beta_0 + \varepsilon \Pi_1(p, 0, 0) = 0, \qquad \alpha_0 + \varepsilon \Pi_2(p, 0, 0) = 0. \tag{23}$$

It follows from this that $\alpha_0 = \varepsilon Q_0,\ \beta_0 = \varepsilon R_0$, where $|Q_0|,\ |R_0| \leqslant M$.

Introduce the change of variables

$$x_1 = x - \alpha_0, \qquad y_1 = y - \beta_0, \tag{24}$$

whence

$$\frac{dx_1}{dt} = \frac{dx}{dt} - \frac{d\alpha_0}{dp}\frac{dp}{dt}, \qquad \frac{dy_1}{dt} = \frac{dy}{dt} - \frac{d\beta_0}{dp}\frac{dp}{dt}. \tag{25}$$

Substitute the right-hand sides of (1) into (25), take (24) into account, expand Π_1, Π_2, Π_3 in Taylor series in powers of x_1, y_1 in a neighborhood of $x_1 = y_1 = 0$, and determine α_1, β_1 from the equations

$$-\beta_1 + \varepsilon \Pi_1(p, \alpha_0, \beta_0) = \varepsilon \Pi_3(p, \alpha_0, \beta_0)\, d\alpha_0/dp,$$

$$\alpha_1 + \varepsilon \Pi_2(p, \alpha_0, \beta_0) = \varepsilon \Pi_3(p, \alpha_0, \beta_0)\, d\beta_0/dp, \tag{26}$$

whence, taking (2), (23) into account, it is seen that $\alpha_1 - \alpha_0 = \varepsilon^2 Q_1,\ \beta_1 - \beta_0 = \varepsilon^2 R_1$, where $|Q_1|,\ |R_1| \leqslant 3M$.

Let $x_n = x - \alpha_{n-1}$, $y_n = y - \beta_{n-1}$. By induction it is easy to show that α_n, β_n must be determined by the equations

$$-\beta_n + \varepsilon\Pi_1(p, \alpha_{n-1}, \beta_{n-1}) = \varepsilon\Pi_3(p, \alpha_{n-1}, \beta_{n-1})\, d\alpha_{n-1}/dp,$$

$$\alpha_n + \varepsilon\Pi_2(p, \alpha_{n-1}, \beta_{n-1}) = \varepsilon\Pi_3(p, \alpha_{n-1}, \beta_{n-1})\, d\beta_{n-1}/dp \tag{27}$$

and that

$$\alpha_n - \alpha_{n-1} = \varepsilon^{n+1} Q_n, \qquad \beta_n - \beta_{n-1} = \varepsilon^{n+1} R_n, \tag{28}$$

where Q_n, R_n are bounded by constants depending only on M, λ.

It is easy to show that, if the series (22) converge, then to $x = \alpha(p)$, $y = \beta(p)$, satisfying (1). We shall show that in the general case the series (22) represent α, β asymptotically, i.e.

$$\alpha - \alpha_{n-1} \sim \varepsilon^n, \qquad \beta - \beta_{n-1} \sim \varepsilon^n \quad \text{for } t \sim 1/\varepsilon. \tag{29}$$

Introduce the variables $r^{(1)}, r^{(2)}$ by the formulas $\alpha = \alpha_{n-1} + r^{(1)}$, $\beta = \beta_{n-1} + r^{(2)}$. Substituting these expressions into (1) in place of x, y and subtracting the corresponding equations (27), taking (28) into account in the latter, we obtain

$$dr^{(1)}/dt + r^{(2)} = \varepsilon P_1, \qquad dr^{(2)}/dt - r^{(1)} = \varepsilon P_2, \tag{30}$$

where

$$P_1 = \Pi_1(p, \alpha_{n-1} + r^{(1)}, \beta_{n-1} + r^{(2)}) - \Pi_1(p, \alpha_{n-1}, \beta_{n-1}) + \varepsilon^n R_n, \tag{31}$$

$$P_2 = \Pi_2(p, \alpha_{n-1} + r^{(1)}, \beta_{n-1} + r^{(2)}) - \Pi_2(p, \alpha_{n-1}, \beta_{n-1}) - \varepsilon^n Q_n.$$

Equations (30) are equivalent to the integral system

$$r^{(1)} = C\cos(t - \gamma) + \varepsilon\cos t \int_0^t (P_1 \cos t + P_2 \sin t)\, dt$$
$$+ \varepsilon \sin t \int_0^t (P_1 \sin t - P_2 \cos t)\, dt,$$

(32)

$$r^{(2)} = C\sin(t-\gamma) + \varepsilon\sin t\int_0^t (P_1\cos t + P_2\sin t)\,dt$$
$$-\varepsilon\cos t\int_0^t (P_1\sin t - P_2\cos t)\,dt,$$

where C, γ are constants expressed in terms of $r^{(1)}(0), r^{(2)}(0)$. We take the latter to be $\lesssim \varepsilon^n$, which is consistent with what follows. In (32) we pass from t to $\tau = \varepsilon t$. We write out the successive Picard approximations, in estimating which we use (2). As a result we obtain that $r^{(1)}, r^{(2)} \sim \varepsilon^n$ for $t \sim 1/\varepsilon$, which, with (28) taken into account, gives (29).

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References

1. V. V. Laricheva, M. V. Rein, *Cosmic Research*, **3**, no. 1, 26 (1965).

2. N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.