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Abstract

Full Text

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PHYSICAL CHARACTERIZATION OF EINSTEIN SPACES OF DEGENERATE TYPE II OF THE PETROV CLASSIFICATION

(Presented by Academician V. A. Fock on 14 X 1964)

1. Let $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ $(\alpha, \beta = 0, 1, 2, 3)$ be the metric of a space-time V_4 of signature -2 (x^0 is the timelike coordinate); $R_{\mu\alpha\beta\nu}$ is the curvature tensor (Riemann-Christoffel); a comma followed by n indices will denote n-fold covariant differentiation. Suppose, further, that $g_{\alpha\beta}(x^\mu)$ are functions of class C^1 (C^3 piecewise). Then the tensor equation

$$\Box\sigma^\sigma R_{\mu\alpha\beta\nu} \equiv g^{\sigma\rho} R_{\mu\alpha\beta\nu,\sigma\rho} = 0 \tag{1}$$

may serve as a generally covariant criterion for the existence of gravitational waves. (This proposition, put forward by A. L. Zelmanov, is substantiated in work (1).)

In the present note we study the spaces V_4 satisfying the tensor wave equation (1), from the point of view of the Petrov classification of gravitational fields according to the algebraic structure of the curvature tensor. The results presented are obtained under the assumption that the given V_4 is an Einstein space $^*T_i$:

$$R_{\alpha\beta} = \varkappa g_{\alpha\beta}, \tag{2}$$

where $i (= 1, 2, 3)$ denotes the number of the corresponding type of gravitational fields. Empty spaces V_4 ($^*T_i$ with $\varkappa = 0$) will be denoted by T_i.

2. Following A. Z. Petrov (2), map the space $^*T_i$ at the given point onto a six-dimensional metrized bivector space R_6, assigning to each skew-symmetric pair of indices $(\alpha\beta)$ and $(\mu\nu)$ of the tensor $R_{\mu\alpha\beta\nu}$ a collective index in the space R_6, using, for example, the numbering:

$$10 \to 1, \quad 20 \to 2, \quad 30 \to 3, \quad 23 \to 4, \quad 31 \to 5, \quad 12 \to 6. \tag{3}$$

As A. Z. Petrov showed (2), the matrix (R_{ab}) $(a, b = 1, 2, 3, 4, 5, 6)$, defining the orthogonal components of the curvature tensor for $^*T_i$, is brought in some orthonormal frame to the following canonical form:

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$$(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \tag{4}$$

where for ${}^*T_1$

$$M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}; \tag{5}$$

$$\sum \alpha_i = -\varkappa, \quad \sum \beta_i = 0; \tag{6}$$

for ${}^*T_2$

$$M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 + 1 & 0 \\ 0 & 0 & \alpha_2 - 1 \end{pmatrix}, \quad N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 1 \\ 0 & 1 & \beta_2 \end{pmatrix}; \tag{7}$$

$$\alpha_1 + 2\alpha_2 = -\varkappa, \quad \beta_1 + 2\beta_2 = 0; \tag{8}$$

for ${}^*T_3$

$$M = \begin{pmatrix} -\varkappa/3 & 1 & 0 \\ 1 & -\varkappa/3 & 0 \\ 0 & 0 & -\varkappa/3 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{9}$$

Let us note the special case of spaces ${}^*T_2$: the degenerate type II (type N ([5])), characterized by the coincidence of eigenvalues of the matrix (R_{ab}):

$$\alpha_1 = \alpha_2 = -\varkappa/3, \qquad \beta_1 = \beta_2 = 0. \tag{10}$$

3. We shall use the well-known result ([3]): in order that the space ${}^*T_i$ satisfy equations (1), it is necessary and sufficient that it satisfy the equations

$$R_{\sigma\alpha\beta}{}^{\delta} R_{\delta\mu\nu}{}^{\sigma} + 2\left(R_{\alpha\nu}{}^{\delta} R_{\mu\beta\delta}{}^{\sigma} - R_{\beta\nu}{}^{\sigma} R_{\mu\alpha\sigma}{}^{\delta} + \varkappa R_{\mu\alpha\beta\nu}\right) = 0. \tag{11}$$

Suppose we have a space ${}^*T_1$. Then, writing equations (11) in the bivector space R_6 in a canonical nonholonomic orthonormal frame, we obtain, using (4), (5), and (6):

$$\alpha_1(\alpha_2 - \alpha_3) - \beta_1(\beta_2 - \beta_3) = 0,$$

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$$\beta_1(\alpha_2 - \alpha_3) + \alpha_1(\beta_2 - \beta_3) = 0$$

and another 4 equations obtained from these by cyclic permutation of the indices 1, 2, 3. These equations determine the necessary and sufficient (see ([4]), p. 120 and ([2]), p. 399) integrability conditions of the equations $R_{\mu\alpha\beta\nu,\sigma} = 0$, defining symmetric spaces, for which equations (1) are satisfied trivially.

Since for $\varkappa = 0$ symmetric Einstein spaces of type I will be flat (see ([2]), p. 402), a space T_1 satisfying condition (1) will always be flat.

Suppose we have a space ${}^*T_2$. Equations (11), with the use of (4) and (7), lead to the system

$$\alpha_1^2 - \beta_1^2 + 2(\alpha_2^2 - \beta_2^2) + \varkappa\alpha_1 = 0,$$

$$2(\alpha_1\beta_1 + 2\alpha_2\beta_2) + \varkappa\beta_1 = 0,$$

$$\alpha_2^2 + 2\alpha_1\alpha_2 - \beta_2^2 - 2\beta_1\beta_2 + 2(\alpha_2 - \alpha_1) + \varkappa(\alpha_2 + 1) = 0,$$

$$\beta_1 - \beta_2 = 0, \tag{12}$$

$$2(\alpha_2\beta_2 + \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_2 - \beta_1) + \varkappa\beta_2 = 0,$$

$$2(\alpha_2 - \alpha_1) + \varkappa = 0,$$

$$\alpha_2^2 + 2\alpha_1\alpha_2 - \beta_2^2 - 2\beta_1\beta_2 + 2(\alpha_1 - \alpha_2) + \varkappa(\alpha_2 - 1) = 0,$$

$$2(\alpha_2\beta_2 + \alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1 - \beta_2) + \varkappa\beta_2 = 0,$$

which, taking (8) into account, can be satisfied only under conditions (10), defining the degenerate type II, with $\varkappa = 0$.

As A. Z. Petrov ([2]) showed, there exists a unique symmetric space ${}^*T_2$; it satisfies conditions (10), which define the degenerate type II for $\varkappa = 0$, and in a special coordinate system is expressed by the metric

$$ds^2 = dx^{0^2} - dx^{1^2} - \operatorname{sh}^2(x^1 \pm x^0)\, dx^{2^2} - \sin^2(x^1 \pm x^0)\, dx^{3^2}. \tag{13}$$

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Thus, type $^*T_2$ admits a unique solution of equations (1), defining a symmetric space–the metric (13).

Finally, let us have the space $^*T_3$. Writing equations (11) in the bivector space with the use of (4) and (9), we ascertain that for any χ they lead to a contradiction. In other words, Einstein spaces satisfying the tensor equation (1) cannot be spaces of type III.

Let us agree to call spaces V_4 with a covariantly constant curvature tensor (symmetric spaces) trivial with respect to equation (1). Then the results set forth can be formulated as the following theorem:

Theorem. *The spaces $^*T_1$, defined by the tensor wave equation (1), can be only trivial with respect to this equation; for $\chi = 0$ (the spaces T_1) they can be only flat. The spaces $^*T_2$, defined by equation (1), including also the space (13) trivial with respect to (1), can belong only to the degenerate type II of the Petrov classification for $\chi = 0$. The spaces $^*T_3$ cannot satisfy equation (1).*

In other words, nonsymmetric Einstein spaces satisfying the wave equation (1) can be only spaces of the degenerate type II T_2.

4. Conversely, putting in formulas (7) $\alpha_i = 0$, $\beta_i = 0$ (the degenerate type II for $\chi = 0$), we ascertain that equations (11) are satisfied identically, i.e., the following holds.

Converse theorem. *Every Einstein space of the degenerate type II for $\chi = 0$ satisfies the generally covariant wave equation (1). Of these, the only space that is trivial with respect to equation (1) is the space with metric (13).*

In conclusion, I consider it a pleasant duty to express my deep gratitude to Prof. A. Z. Petrov for valuable critical remarks.

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Note: Figure translations are in progress. See original paper for figures.

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